Anti-cyclotomic p-adic interpolation of definite Friedberg–Jacquet integrals

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Abstract. Let F/F^+ be a CM extension, p a fixed prime such that every $\mathfrak{p}|p$ in F^+ splits in F and π a regular algebraic cuspidal automorphic representation of a definite unitary group of rank $2n$. For suitable standard parabolics and parabolic non-critical regular refinements of π at p, we construct: (A) finite-slope p-adic distributions on the anti-cyclotomic ray class group which interpolate associated Friedberg–Jacquet integrals, (B) neighbourhoods of the parabolic eigenvariety étale over the weight space, and (C) two-variable distributions that interpolate the finite-slope distributions over such neighbourhoods.

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1 Introduction

1.1 Motivation

1.1.1 Context

According to conjectured explicit reciprocity laws, there are two equivalent ways to construct p-adic L-functions: p-adic interpolation of period integrals and the big logarithm machinery of Perrin-Riou. The resulting p-adic L-functions then play a central role by appearing directly in Iwasawa main conjectures and often indirectly in the Bloch–Kato conjecture via Euler systems. In this framework, period integrals serve as the main bridge between the cohomology that is used to encode the L-functions and the special values of the L-functions themselves.

The aim of this article is to further the ideas of Bertolini–Darmon–Prasanna as they have appeared in [BDP13] to study the arithmetic in the anti-cyclotomic direction by building a distribution that interpolates the unitary FJ-period, this being conjectured to give the square-roots of special L-values.

We do this through the over-convergent construction method conceived by Pollack–Stevens [PS11] and since developed in numerous articles. Regarding the last, the author draws particular inspiration from [BDGJW25], [SDW24], [Wil23] and [Dim24]. Furthermore, this article is closely related to that of Graham in [Gra24] and [Gra26] where a parallel case of certain indefinite unitary groups of rank $2n$ is studied.

1.1.2 Applications

The interpolated definite Friedberg–Jacquet period integrals are conjectured to be deeply connected to special values of L-functions. If we let F/F^+ be a CM extension and $\mathbf{U}_N$ denote definite unitary group of rank N, then we have a conjecture:

Conjecture 1.1 ([CG22, Conjecture 7.4])**.** *Let π be a tempered cuspidal automorphic representation of $\mathbf{U}_{2n}$ with central character ω_π such that for some Hecke character $\tilde{\eta} : F^\times \mathbb{A}_{F^+}^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}$ we have $\omega_\pi(z) = \tilde{\eta}(\overline{z}/z)^n$ and $\chi : F^\times \mathbb{A}_{F^+}^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}$ an algebraic anti-cyclotomic Hecke character. Then:*

$$\mathscr{P}^\infty_{\mathbf{H},\chi',\eta}(\varphi) := \int_{[\mathbf{H}]} \eta(\det h_2^{-1})\chi'\left(\frac{\det(h_2)}{\det(h_1)}\right)\varphi(h)dh \neq 0. \tag{1.1}$$

where $\mathbf{H} := diag(\mathbf{U}_n \times \mathbf{U}_n)$, $[\mathbf{H}] := \mathbf{H}(F^+)\backslash\mathbf{H}(\mathbb{A}_{F^+})$, dh is the Tamagawa measure and χ' and η are the unique characters such that $\chi'(z) = \chi(\overline{z}/z)$ and $\eta(z) = \tilde{\eta}(\overline{z}/z)$ is equivalent to the following conditions:

1. *$BC(\pi)$ is of symplectic similitude type with similitude character η,*
2. *$\pi \otimes \chi$ is locally distinguished, $\mathrm{Hom}_{\mathbf{H}(\mathbb{A}_{\mathbb{Q},f})}(\pi_f, \chi' \otimes (\chi')^{-1}) \neq 0$, and*
3. *$L(BC(\pi) \boxtimes \chi, 1/2) \neq 0$ for the standard L-function.*

In light of recent work by Leslie–Xiao–Zhang [LXZ25a] and [LXZ25b, Proposition 5.9. and 6.17.], it seems reasonable that for certain $\varphi = \otimes_v \varphi_v \in \pi$ one can expect an Ichino–Ikeda style formula:

$$\frac{|\mathscr{P}^\infty_{\mathbf{H},\chi,\eta}(\varphi)|^2}{\langle \varphi, \varphi \rangle} = (*) \cdot L(\mathrm{BC}(\pi) \boxtimes \chi, 1/2) \cdot \prod_v I_v(\varphi_v) \tag{1.2}$$

for some factor $(*)$ and certain normalised matrix coefficients $I_v(\varphi_v)$ defined by [LXZ25b] such that $I_v(\varphi_v) = 1$ for all unramified π_v. This should make our construction and that of Graham into genuine (square-root) p-adic L-functions. Our construction will be related to $L(\mathrm{BC}(\pi) \boxtimes \chi, 1/2)$ with sign $+1$ in the functional equation unlike Graham's construction for sign -1. It is anticipated to contribute towards the Iwasawa main conjecture via a generalisation of [BD05] using the bipartite Euler system formalism of [How06] as in Corato-Zanarella's proof of the first reciprocity law [Cor26].

1.1.3 Summary

Because of the involved prerequisites needed to state the main theorems with complete rigour, we first give a rough indicative summary of the results to motivate and give context to the preliminaries. Then we dedicate the rest of the introduction to give the necessary definitions and the exact statements. Let $\pi = \otimes \pi_v$ be a regular algebraic cuspidal[1] automorphic representation of $\mathbf{U}_{2n}(\mathbb{A}_{F^+})$ and choose a standard parabolic $\mathbf{P}$ of $\mathbf{GL}_{2n}(\mathbb{Q}_p)$ contained in the (n,n)-parabolic. As is standard, we assume $\tilde{\pi} = (\pi, \alpha_p)$ is a non-$\mathbf{P}$-critical regular $\mathbf{P}$-parahoric refinement. Lastly, we assume $K_f = K^p \mathrm{Iw}_{\mathbf{P}} \subseteq \mathbf{U}_{2n}(\mathbb{A}_{F^+,f})$ is an open compact, $\pi_{f,E}$ is a E-model for some number field E and $\varphi_f \in \tilde{\pi}_{f,E}^{K_f}$ (See remark 1.2). Then:

(A) There exists a locally analytic distribution $\mu \in \mathcal{D}(\Gamma^{\mathrm{ac}}, L)$ of growth $v_p(\alpha_{\mathbf{P},p})$ such that for anti-cyclotomic Hecke characters χ with conductor $p^\beta\mathfrak{R}$ we have for a non-zero factor $(*)$:

$$\mu(\varphi)(\chi) = (*) \cdot \mathscr{P}^\infty_{\mathbf{H},\chi,\eta}(v_{\underline{\lambda}} \otimes \varphi_f). \tag{1.3}$$

for a fixed $v_{\underline{\lambda}} \in \pi_\infty$.

[1] π is automatically cuspidal due to compactness at infinity. See for example [Loe08, Page 9]

(B) The weight map $w : \mathscr{E}^{\mathbf{P}} \to \mathscr{W}_K^{\mathbf{P}}$ is étale at $\tilde{\pi} \in \mathscr{E}^{\mathbf{P}}$.

(C) For a neighbourhood $\Omega \subset \mathscr{W}_K^{\mathbf{P}}$ with a Zariski-dense subset of classical points with rational models Ω_{scl} from part (B) where $w(\tilde{\pi}) \in \Omega$, there exists a unique distribution $\mu_\Omega \in \mathcal{D}(\Gamma^{\mathrm{ac}}, \mathcal{O}_\Omega)$ such that for all $\underline{\lambda} \in \Omega_{\mathrm{scl}}$, there exists $c_{\underline{\lambda}} \in L^\times$ such that

$$\mathrm{sp}_{\underline{\lambda}}(\mu_\Omega(\varphi)) = c_{\underline{\lambda}} \mu(\varphi_{\underline{\lambda}}) \tag{1.4}$$

where $\varphi_{\underline{\lambda}} \in \tilde{\pi}_{\underline{\lambda}}$ is in an automorphic representation corresponding to $\underline{\lambda}$ via $w(\tilde{\pi}_{\underline{\lambda}}) = \underline{\lambda}$.

Remark 1.2. *It is expected that the test vector should have the form*

$$\varphi_f = \varphi_p \otimes \left(\bigotimes_{v \in S} \varphi_v \right) \otimes \left(\bigotimes_{v \notin S \cup \{p\}} \varphi_v \right) \tag{1.5}$$

where S is a finite set of bad primes which include v such that π_v is ramified. If there exists K_f such that $\dim \tilde{\pi}_f^{K_f} = 1$, which seems plausible considering the $\mathbf{GL}_{2n}$ case as exemplified in [JPS81], then it is trivial to prove $\Omega\varphi \in \pi_{f,E}$. However, if $\dim \tilde{\pi}_f^{K_f} > 1$, then the dimensionality issue lies in S with ramified representations. In these cases, we find a distinguished line $L_v \subseteq \pi_v^{K_v}$ for $v \in S$ that is compatible with a rational structure. The construction of a p-adic L-function is an immediate consequence once an Ichino-Ikeda formula has been proved, and the vectors at $u^{-1} t_{p,\mathbf{P}}^{\beta} \varphi_p$ are already calculated in [BDGJW25].

Before we continue, it is worth emphasising a novelty, that being the ability to choose the parabolic. This has a direct implication in the form of a trade-off:

Example 1.3. *Let $\mathbf{P}_k$ be the parabolic with Levi subgroup*

$$\mathbf{L}_{\mathbf{P}_k} = \mathbf{GL}_k \times \mathbf{GL}_1 \times ... \times \mathbf{GL}_1 \times \mathbf{GL}_k \tag{1.6}$$

then we can see that for bigger k the non-$\mathbf{P}$-critical assumption becomes weaker, with for example $v_p(\alpha_{\mathbf{P}_{k+1},p}) \leq v_p(\alpha_{\mathbf{P}_k,p})$. However, the weight space gets smaller because $\dim \Omega = [F^+ : \mathbb{Q}](n-k+1)+1$ so we get less information regarding the eigenvariety.

1.2 Notation

1.2.1 The global and local algebra

For number fields F, let $\Sigma_F := \{\sigma : F \hookrightarrow \overline{\mathbb{Q}}\}$ be the embeddings, $\mathbb{A}_F$ the adeles and $\mathbb{A}_{F,f}$ the finite adeles. From now, F/F^+ is a CM field with $\mathrm{Gal}(F/F^+) = \langle c \rangle$ and p is a fixed prime such that $\mathfrak{p}|p$ in F split in F. We fix inclusions $\overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}}_p$, $\overline{\mathbb{Q}} \subseteq \mathbb{C}$, $L/\mathbb{Q}_p$ a finite extension such that $\forall \sigma \in \Sigma_F$ we have $\sigma(F) \subseteq L$ and for each $\mathfrak{p}|p$ an embedding $\sigma_{\mathfrak{p}} : F_{\mathfrak{p}} \hookrightarrow L$. For every finite place ν of F, we pick a uniformiser ϖ_ν of $\mathcal{O}_{F_\nu}$. Lastly, for compact subsets $X \subseteq \mathbb{Q}_p^r$ and $\mathbb{Q}_p$-algebras R, and we denote locally analytic functions $f : X \to R$ as $\mathcal{A}(X, R)$ and the dual distributions as $\mathcal{D}(X, R) := \mathrm{Hom}_{\mathrm{cts}}(\mathcal{A}(X, R), R)$.

1.2.2 Anti-cyclotomic theory

For anti-cyclotomic parts, we will consider the admissible algebraic anti-cyclotomic Hecke characters

$$\chi : \Gamma^{\mathrm{ac}} \to \mathbb{C}^\times \text{ where } \Gamma^{\mathrm{ac}} := \mathbb{A}_F^\times / F^\times \mathbb{A}_{F^+}^\times. \tag{1.7}$$

and we say they have infinity type $(\underline{j}, -\underline{j})$ where $\underline{j} := (j_\sigma)_{\sigma \in \Sigma_{F^+}}$ for $\chi_\infty(z) = \prod_{\sigma_\sigma \in \Sigma_{F^+}} \sigma(z)^{j_\sigma} c(\sigma(z))^{-j_\sigma}$ and conductor $\mathfrak{R}p^\beta$ in $\mathcal{O}_F$ where $p^\beta := \prod_{\mathfrak{p}|p} \mathfrak{p}^{\beta_{\mathfrak{p}}}$ and $\mathfrak{R}$ are co-prime ideals of $\mathcal{O}_{F,+}$. We then define define the torus

$$\mathbf{T}_1 := \ker(\mathrm{Res}_{F/F^+} \mathbf{G}_m \xrightarrow{\mathrm{Nm}_{F/F^+}} \mathbf{G}_m) \tag{1.8}$$

for the norm Nm_{F/F^+} and observe that we have the corresponding Hecke character $\chi' : \mathbf{T}_1(F^+) \backslash \mathbf{T}_1(\mathbb{A}_{F^+}) \to \mathbb{C}$ such that $\chi'(z) = \chi(c(z)/z)$.

1.2.3 Locally analytic Hecke characters

To enable p-adic variation of arithmetic Hecke characters χ, we associate locally analytic characters $\chi_{[p]} : \mathrm{Gal}(F_{p^\infty\mathfrak{R}}/F) \to L$ where $F_{p^\infty\mathfrak{R}}$ is the maximal anti-cyclotomic extension unramified outside $p^\infty\mathfrak{R}$. By global Artin reciprocity, we have that

$$\mathrm{Gal}(F_{p^\infty\mathfrak{R}}/F) \cong \Gamma^{\mathrm{ac}}_{\mathfrak{R}p^\infty} := \mathbf{T}_1(F^+)\backslash\mathbf{T}_1(\mathbb{A}_{F^+,f})/(\mathbf{T}_1(\mathbb{A}^p_{F^+,f}) \cap (1+\mathfrak{R}\widehat{\mathcal{O}}^p_F)). \tag{1.9}$$

It then follows that we have a natural map $j : \mathbf{T}_1(\mathcal{O}_{F^+}\otimes\mathbb{Z}_p) \to \Gamma^{\mathrm{ac}}_{p^\infty\mathfrak{R}}; t \mapsto (..,1,t,1,..)$ with finite kernel and co-kernel. Similar to [Wil23, Definition 2.7.], we then have through the j map

$$\chi_{[p]}(x) := \chi(x)\cdot(\chi|_{(F^+\otimes_{\mathbb{Q}}\mathbb{R})^\times}(x))^{-1}\cdot\prod_{\sigma\in\Sigma_F}\sigma(x)^{j_\sigma} \text{ where } \sigma : F^+\otimes_{\mathbb{Z}}\mathbb{Z}_p \to L \text{ restricts to } \sigma\in\Sigma_{F^+}. \tag{1.10}$$

1.2.4 Hermitian spaces and algebraic groups

Suppose V is a definite Hermitian space of dimension $2n$ over F and W_1, W_2 definite Hermitian spaces of dimension n such that $V = W_1\oplus W_2$. We then define the two main groups of the article as

$$\mathbf{H} := \mathrm{Res}_{F^+/\mathbb{Q}}(\mathbf{U}(W_1)\times\mathbf{U}(W_2)) \subseteq \mathbf{G} := \mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{U}(V) \tag{1.11}$$

where $\mathbf{U}(-)$ denotes the associated Hermitian group. Note $\mathbf{G}(\mathbb{Q}_p) = \mathbf{GL}(V)(F^+\otimes\mathbb{Q}_p)$ and similarly for $\mathbf{H}$ because $\mathfrak{p}|p$ split in F, and hence we have canonical subgroups $\mathbf{G}(\mathbb{Z}_p)$ and $\mathbf{H}(\mathbb{Z}_p)$. For parabolic subgroups $\mathbf{P}\subseteq\mathbf{G}$, we let $\mathbf{N_P}$ and $\mathbf{L_P}$ be the associated unipotent and Levi subgroups, and for subgroups $\mathbf{M}\subseteq\mathbf{G}$, we denote the restriction as $\mathbf{P_M} := \mathbf{P}\cap\mathbf{M}$ and $\mathbf{M}^-$ as the opposite. We define standard parabolics $\mathbf{P}\subseteq\mathbf{GL}_{2n,F^+}(V)$ to be *relevant* if and only if

$$u := \begin{pmatrix} I_n & w_n \\ 0 & I_n\end{pmatrix} \in \mathbf{P}(F^+) \tag{1.12}$$

where w_n is the longest Weyl element of $\mathbf{GL}_{n,F}(W_2)$. Lastly, let $\mathbf{T}_{2n}\subseteq\mathbf{B}$ be the diagonal torus and upper triangular matrices, and fix $\mathbf{T} := \mathrm{Res}_{F/\mathbb{Q}}\mathbf{T}_{2n}$.

1.2.5 Algebraic and analytic weight spaces

Using [Mil17, Paragraph 2.61], we denote the dominant weights $X^*_+(\mathbf{T}) \cong \prod_{\sigma\in\Sigma_F}X^*_+(\mathbf{T}_{2n})$ described as tuples of integers $(\lambda_{1,\sigma},..,\lambda_{2n,\sigma})$ with $\lambda_{1,\sigma}\geq..\geq\lambda_{2n,\sigma}$ and define the individually pure weights as

$$X^*_0(\mathbf{T}) := \{\underline{\lambda} = (\lambda_\sigma)_{\sigma\in\Sigma_F}\in X^*_+(\mathbf{T}) : \forall\sigma\in\Sigma_{F^+}\exists w_\sigma\in\mathbb{Z} \text{ s.t } \forall j\ \lambda_{\sigma,j}+\lambda_{\sigma,2n+1-j} = w_\sigma\}. \tag{1.13}$$

If $K_f\subseteq\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})$ is an open compact subgroup and $\mathbf{P}$ a relevant parabolic, then we define the weight space as $\mathscr{W}_{K_f}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}} = \mathrm{Spf}\left(\mathbb{Z}_p[[\mathbf{T}(\mathbb{Z}_p)/\overline{Z}_{K_f}]]\right)^\#$ where $\overline{Z}_{K_f}$ is the p-adic closure of $Z_{\mathbf{G}}(\mathbb{Q})\cap K_f$ with $Z_{\mathbf{G}}$ being the centraliser and $\#$ meaning that we take the generic fiber. With this rigid analytic space, we can take the $L/\mathbb{Q}_p$ points

$$\mathscr{W}_{K_f}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}(L) := \mathrm{Hom}_{\mathrm{cts}}(\mathbf{L}_{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}(\mathbb{Z}_p)/\overline{Z}_{K_f}, L^\times). \tag{1.14}$$

Then for rigid analytic affinoids $\Omega\subseteq\mathscr{W}_{K_f}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}(L)$, we have a coordinate ring $\mathcal{O}_\Omega$ and an associated character character $\chi_\Omega : \mathbf{T}(\mathbb{Z}_p)\to\mathcal{O}_\Omega$ with specialisation maps such that for all classical $\underline{\lambda}\in\Omega^{cl} := X^*_+(\mathbf{T})\cap\Omega$ we have $\underline{\lambda} = \mathrm{sp}_{\underline{\lambda}}\circ\chi$. We also define the pure weight space as

$$\mathscr{W}_{K_f,0}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}(L) := \overline{X^*_0(\mathbf{T})\cap\mathscr{W}_{K_f}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}(L)} \qquad \text{(Zariski closure)}. \tag{1.15}$$

1.2.6 The parahoric subgroups

We define the parahoric at a parabolic $\mathbf{P} \subseteq \mathbf{GL}_{2n}$ to be $\mathrm{Iw}_{\mathbf{P}} := \prod_{\mathfrak{p}|p} \mathrm{Iw}_{\mathbf{P},\mathfrak{p}}$ where

$$\mathrm{Iw}_{\mathbf{P},\mathfrak{p}} := \{g \in \mathbf{GL}_{2n}(\mathcal{O}_{F_{\mathfrak{p}}}) : g \mod \mathfrak{p} \in \mathbf{P}(\mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p})\}. \tag{1.16}$$

Such subgroups have a natural parahoric decomposition

$$\mathrm{Iw}_{\mathbf{P},\mathfrak{p}} = (\mathrm{Iw}_{\mathbf{P},\mathfrak{p}} \cap \mathbf{N}_{\mathbf{P}}^{-}(\mathcal{O}_{F_{\mathfrak{p}}})) \cdot L_{\mathbf{P}}(\mathcal{O}_{F_{\mathfrak{p}}}) \cdot N_{\mathbf{P}}(\mathcal{O}_{F_{\mathfrak{p}}}). \tag{1.17}$$

1.2.7 Hecke operators

For each $\mathfrak{p}|p$ and relevant parabolic $\mathbf{P} \subseteq \mathbf{GL}_{2n}$ with Levi $\mathbf{L}_{\mathbf{P}} \cong \mathbf{GL}_{m_1} \times ... \times \mathbf{GL}_{m_r}$, we define:

$$t_{\mathbf{P},\mathfrak{p}} := \mathrm{diag}(\varpi_{\mathfrak{p}}^{r-1} I_{m_1}, ..., \varpi_{\mathfrak{p}} I_{m_{r-1}}, I_{m_r}) \tag{1.18}$$

Given an open compact subgroup $K = \prod_v K_v \subseteq \mathbf{G}(\mathbb{A}_{\mathbb{Q}})$, we define $U_{\mathbf{P},\mathfrak{p}} := [K_{\mathfrak{p}} t_{\mathbf{P},\mathfrak{p}} K_{\mathfrak{p}}]$. Let S be a set of bad primes of F containing $\mathfrak{p}|p$ and the places v such that π_v is not spherical and K_v not hyperspecial maximal compact. For $v \notin S$ and $\nu \in X_+^*(\mathbf{T})$, we define $T_{v,\nu} := [K_v \nu(\varpi_v) K_v]$. We then define the Hecke algebras as

$$\mathcal{H}_p^{\mathbf{P}} := \mathbb{Z}[U_{\mathbf{P},\mathfrak{p}} : \mathfrak{p}|p],\ \mathcal{H}^{'S} := \mathbb{Z}[T_{v,\nu} : \nu \in X_+^*, v \notin S] \text{ and } \mathcal{H}_{\mathbf{P}}^S := \mathcal{H}_p \otimes_{\mathbb{Z}} \mathcal{H}^{'S}. \tag{1.19}$$

1.2.8 Representation theory

Let R be a sufficiently large étale $\mathbb{Q}_p$-algebra. For any $\underline{\lambda} \in X_+^*(\mathbf{T}_R)$, we denote the corresponding R-valued irreducible $\mathrm{Res}_{F^+/\mathbb{Q}} \mathbf{GL}_{2n}$-representation of highest weight $\underline{\lambda}$ as $V_{\underline{\lambda}}(R)$. We note that

$$\mathbf{G}_R \cong (\mathrm{Res}_{F^+/\mathbb{Q}} \mathbf{GL}_{2n})_R \text{ and } \mathbf{H}_R \cong (\mathrm{Res}_{F^+/\mathbb{Q}} \mathbf{GL}_n \times \mathbf{GL}_n)_R. \tag{1.20}$$

A consequence of this is that we may view R-valued representations of $\mathbf{G}$ for subalgebras $E \subseteq R$ as the representations of $(\mathrm{Res}_{F^+/\mathbb{Q}} \mathbf{GL}_{2n})_R$ because the R-valued representations of groups $\mathbf{G}(E)$ factor through $\mathbf{G}(R)$ pointwise, and likewise for $\mathbf{H}$. Lastly, we define for each $\underline{j}_i = (j_{\sigma,i})_{\sigma \in \Sigma_{F^+}}$ the representation $V_{(\underline{j}_1, \underline{j}_2)}$ of $\mathbf{H}$ via the decompositon $\mathbf{H}_L \cong \prod_{\sigma \in \Sigma_{F^+}} (\mathbf{GL}_n \times \mathbf{GL}_n)_{\sigma,L}$ corresponding to the function

$$h = (h_{\sigma,1}, h_{\sigma,2})_{\sigma_{\sigma \in \Sigma_{F^+}}} \mapsto \prod_{\sigma \in \Sigma_{F^+}} \det(h_{\sigma,1})^{j_{\sigma,1}} \det(h_{\sigma,2})^{j_{\sigma,2}}. \tag{1.21}$$

1.2.9 Coefficient modules

Following [SW21, Section 3.2.3], let $\mathbf{P}$ be a parabolic and M be a finite Banach R-module with left $L_{\mathbf{P}}(\mathbb{Z}_p)$ action extended to $\mathrm{Iw}_{\mathbf{P}}^{-} L_{\mathbf{P}}(\mathbb{Z}_p)$ via trivial $\mathrm{Iw}_{\mathbf{P}} \cap \mathbf{N}_{\mathbf{P}}^{-}(\mathbb{Z}_p)$ action. We then define locally analytic induction

$$\mathrm{LAInd}_{\mathbf{L}_{\mathbf{P}}(\mathbb{Z}_p)}^{\mathrm{Iw}_{\mathbf{P}}}(M) := \{f \in \mathcal{A}(\mathrm{Iw}_{\mathbf{P}}, M) : \forall (g,b) \in \mathrm{Iw}_{\mathbf{P}} \times \mathrm{Iw}_{\mathbf{P}}^{-} L_{\mathbf{P}}(\mathbb{Z}_p),\ f(bg) = b * f(g)\}. \tag{1.22}$$

Analogous to the Borel–Weil–Bott theorem, we then define certain coefficient modules

$$\mathcal{A}_{\Omega}^{\mathbf{P}} := \mathrm{LAInd}_{\mathbf{L}_{\mathbf{P}}(\mathbb{Z}_p)}^{\mathrm{Iw}_{\mathbf{P}}}(\mathrm{Ind}_{\mathbf{B}(\mathbb{Z}_p)}^{\mathbf{L}_{\mathbf{P}}(\mathbb{Z}_p)} \underline{\lambda} \otimes \mathcal{O}_{\Omega}) \text{ and } \mathcal{D}_{\Omega}^{\mathbf{P}} := \mathrm{Hom}_{\mathrm{cts}}(\mathcal{A}_{\Omega}^{\mathbf{P}}, \mathcal{O}_{\Omega}). \tag{1.23}$$

As defined in [SW21, Remark 4.11], for parabolics $\mathbf{P} \subseteq \mathbf{P}'$, we have an injection and hence induced projection

$$\iota_{\mathbf{P}'/\mathbf{P}} : \mathcal{A}_{\Omega}(\mathrm{Iw}_{\mathbf{P}'}) \hookrightarrow \mathcal{A}_{\Omega}(\mathrm{Iw}_{\mathbf{P}}) \text{ and } p_{\mathbf{P}'/\mathbf{P}} : \mathcal{A}_{\Omega}(\mathrm{Iw}_{\mathbf{P}}) \twoheadrightarrow \mathcal{A}_{\Omega}(\mathrm{Iw}_{\mathbf{P}'}) \tag{1.24}$$

with corresponding dual maps $\iota_{\mathbf{P}'/\mathbf{P}}^{\vee}$ and $p_{\mathbf{P}'/\mathbf{P}}^{\vee}$ for the distribution coefficient modules $\mathcal{D}_{\Omega}^{\mathbf{P}}$.

1.2.10 Hecke actions on coefficient modules

One may observe that the parahoric decomposition gives $\mathcal{A}(\mathbf{N}_{\mathbf{P}}(\mathbb{Z}_p), M) \cong \mathrm{LAInd}_{\mathbf{L}_{\mathbf{P}}(\mathbb{Z}_p)}^{\mathrm{Iw}_{\mathbf{P}}}(M)$, and this permits us to define for $f \in \mathcal{A}_{\Omega}^{\mathbf{P}}$ an action

$$(t_{\mathbf{P},\mathfrak{p}}^{-1} * f)(n) := f(t_{\mathbf{P},\mathfrak{p}} n t_{\mathbf{P},\mathfrak{p}}^{-1}) \tag{1.25}$$

and this extends to distributions via $(t_{\mathbf{P},\mathfrak{p}} * \mu)(f) := \mu(t_{\mathbf{P},\mathfrak{p}} * f)$. Notably, this induces an action on $V_{\underline{\lambda}} \cong \mathcal{A}_{\underline{\lambda}}^{\mathbf{G}}$ different from the standard algebraic action $t_{\mathbf{P},\mathfrak{p}} \cdot v$. Importantly, these actions are compatible

$$t_{\mathbf{P},\mathfrak{p}} * \mu = \underline{\lambda}(t_{\mathbf{P},\mathfrak{p}}) \times (t_{\mathbf{P},\mathfrak{p}} \cdot \mu). \tag{1.26}$$

This factors down to any Hecke operators, and we will denote the operators arising from this action on the coefficient modules $\mathcal{D}_{\Omega}^{\mathbf{P}}$ as $U_{\mathbf{P},\mathfrak{p}}^{\circ}$ and any eigenvalues as $\alpha_{\mathbf{P},\mathfrak{p}}^{\circ}$.

1.2.11 Automorphic representations and critical L-values

We consider RACARs (regular algebraic automorphic representations) $\pi \subset L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{Q}}))$ of $\mathbf{G}$, and they are both automatically cuspidal and cohomological [BG14, Section 8.3] in the sense of [GH24, Definition 15.6], that is we associate a unique weight $\underline{\lambda}_{\pi}$ such that it contributes non-trivially to

$$\mathrm{H}^{\bullet}(\mathfrak{g}_{\infty}, K_{\infty}^{\circ}; \pi_{\infty} \otimes V_{\underline{\lambda}_{\pi}}^{\vee}(\mathbb{C})) \neq 0. \tag{1.27}$$

With the weight $\underline{\lambda}_{\pi}$, there is a concrete description of the infinity types $\underline{j}$ for the admissible Hecke characters χ' such that $L(\pi \times \chi', 1/2)$ is a Deligne-critical value given by

$$\mathrm{Crit}(\underline{\lambda}_{\pi}) := \{(j_{\sigma})_{\sigma \in \Sigma_{F^+}} \in \mathbb{Z}^{\Sigma_{F^+}} : \forall \sigma \in \Sigma_{F^+} \ -\lambda_{\sigma,n} \leq j_{\sigma} \leq -\lambda_{\sigma,n+1}\}. \tag{1.28}$$

We then recall that $L(1/2, \pi \times \chi')$ is a Deligne-critical value if and only if $L(1/2, \mathrm{BC}_F(\pi) \times \chi)$ is a Deligne-critical value for the corresponding anti-cyclotomic χ with infinity type $(j_{\sigma}, -j_{\sigma})_{\tau}$ for $\tau \in \Sigma_F$ where $\tau|_{F^+} = \sigma$. We abuse notation and let $\mathrm{Crit}(\underline{\lambda})[p^{\beta}\mathfrak{R}]$ denote the corresponding anti-cyclotomic Hecke characters of conductor $\mathfrak{R}p^{\beta} := \mathfrak{R}\prod_{\mathfrak{p}|p} \mathfrak{p}^{\beta_{\mathfrak{p}}}$. Lastly, we define $\delta_{\mathbf{B}} : \mathbf{T}(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}^{\times}$ to be the standard modulus character.

1.2.12 Refinement and generalised eigensystems

Let $\mathbf{P}$ be a parabolic of $\mathbf{G}$. Given a choice of an $U_{\mathbf{P},\mathfrak{p}}$-eigenvalue $\alpha_{\mathfrak{p}}$ on $\pi_{\mathfrak{p}}$, we call $(\pi_{\mathfrak{p}}, \alpha_{\mathfrak{p}})$ a $\mathbf{P}$-refinement at $\mathfrak{p}|p$, and we say it is regular if $\alpha_{\mathfrak{p}}$ is a simple eigenvalue. We define $\tilde{\pi} := \{(\pi_{\mathfrak{p}}, \alpha_{\mathfrak{p}}) : \mathfrak{p}|p\}$ to be $\mathbf{P}$-refinement, and we say $\tilde{\pi}$ is regular when each $(\pi_{\mathfrak{p}}, \alpha_{\mathfrak{p}})$ is regular. Given a $\mathbf{P}$-refinement $\tilde{\pi}$, we have a homomorphism:

$$\psi_{\tilde{\pi}} : \mathcal{H} \otimes F^+ \to F^+ \tag{1.29}$$

where ψ_{π} sends $T_{\nu,v}$ to its eigenvalue on the line $\pi_v^{K_v}$ and $U_{\mathbf{P},\mathfrak{p}}$ to $\alpha_{\mathfrak{p}}$. If M is an L-vector space with a $\mathcal{H}$ action, we denote the localisation of M at $\ker \psi_{\tilde{\pi}}$ as $M_{\tilde{\pi}}$, and if M is finite dimensional, then it is the generalised eigenspace for $\mathcal{H} \otimes L$.

1.2.13 Local systems and parabolic criticality

Suppose M is a left K-module. Then we associate to M the local system on the locally symmetric space $Y_K := \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{Q}})/KK_{\infty}^{\circ}$ given by locally constant sections of $\mathbf{G}(\mathbb{Q})\backslash[\mathbf{G}(\mathbb{A}_{\mathbb{Q}}) \times M]/KK_{\infty}^{\circ}$ with action $\gamma(g,m)kz = (\gamma gkz, k^{-1} \cdot m)$. We let $\mathrm{H}_c^{\bullet}(Y_K, -)$ be compactly supported Betti cohomology, noting that since Y_K is finite, we have $\mathrm{H}_c^{\bullet} = \mathrm{H}^{\bullet}$ and only H^0 is non-trivial, and observe that equation 1.24 induces

$$\iota_{\mathbf{G}/\mathbf{P}}^{\vee} : \mathrm{H}_c^{\bullet}(Y_K, \mathscr{D}_{\lambda}^{\mathbf{P}}) \to \mathrm{H}_c^{\bullet}(Y_K, \mathscr{V}_{\lambda}). \tag{1.30}$$

We define $\tilde{\pi}$ to be non-$\mathbf{P}$-critical at slope $h \in \mathbb{Q}_{>0}$ and open compact subgroup K when the localisation at the generalised eigenspace give an isomorphism:

$$\iota_{\mathbf{G}/\mathbf{P}}^{\vee} : \mathrm{H}_c^{\bullet}(Y_K, \mathscr{D}_{\lambda}^{\mathbf{P}})_{\tilde{\pi}}^{\leq h} \to \mathrm{H}_c^{\bullet}(Y_K, \mathscr{V}_{\lambda})_{\tilde{\pi}}^{\leq h}. \tag{1.31}$$

1.2.14 The parabolic eigenvaritety

Let $\mathbf{P}$ be a relevant parabolic, $K \subseteq \mathbf{G}(\mathbb{A}_{\mathbb{Q}})$ an open compact subgroup and $\Omega \subseteq \mathscr{W}_{K_f}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}$ an affinoid. For $h \in \mathbb{Q}_{>0}$, we then say Ω is h-adapted if there is a slope decomposition

$$\mathrm{H}_c^\bullet(Y_K, \mathcal{D}_\Omega^{\mathbf{P}}) = \mathrm{H}_c^\bullet(Y_K, \mathcal{D}_\Omega^{\mathbf{P}})^{\leq h} \times \mathrm{H}_c^\bullet(Y_K, \mathcal{D}_\Omega^{\mathbf{P}})^{>h}. \tag{1.32}$$

We then define the local eigenalgebra and eigenvariety respectively as

$$\mathbb{T}_{\Omega,h}^{\mathbf{P}} := \mathrm{Im}(\mathcal{H}_{\mathbf{P}}^S \otimes \mathcal{O}_\Omega \to \mathrm{End}_{\mathcal{O}_\Omega}(\mathrm{H}_c^\bullet(Y_K, \mathcal{D}_\Omega^{\mathbf{P}})^{\leq h})) \text{ and } \mathscr{E}_{\Omega,h}^{\mathbf{P}} := \mathrm{Sp}(\mathbb{T}_{\Omega,h}^{\mathbf{P}}). \tag{1.33}$$

Via the natural map $\mathcal{O}_\Omega \to \mathbb{T}_{\Omega,h}^{\mathbf{P}}$, we define the weight map

$$w : \mathscr{E}_{\Omega,h}^{\mathbf{P}} \to \mathcal{O}_\Omega. \tag{1.34}$$

Gluing these spaces and maps together, we get $\mathscr{E}^{\mathbf{P}}$ and $w : \mathscr{E}^{\mathbf{P}} \to \mathscr{W}_{K_f}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}$. Importantly, the L-points $y \in \mathscr{E}^{\mathbf{P}}$ biject with the systems of eigenvalues $\psi_y : \mathcal{H}^\circ \to L$ appearing in $\mathrm{H}_c^\cdot(Y_K, \mathcal{D}_\Omega^{\mathbf{P}})^{\leq h}$ where the $\circ$ indicates that Hecke algebra acts via $U_{\mathbf{P},p}^\circ$. Lastly, we define classical points in $\mathscr{E}^{\mathbf{P}}$ as those corresponding to $\psi_{\tilde{\pi}}$ for some automorphic representation π of $\mathbf{G}(\mathbb{A}_{\mathbb{Q}})$ as in equation 1.29.

1.2.15 Growth of locally analytic distributions

Following an adapted version of [Loe14] as in [Wil23, secction 7.2], we let $\underline{m} \in \mathbb{Z}_{\geq 1}^{\mathfrak{p}|p}$ and observe that we have an exact sequence

$$0 \to \mathscr{U}_{\underline{m}} \hookrightarrow \Gamma_{p^\infty\mathfrak{R}}^{\mathrm{ac}} \to \mathrm{Cl}_{\mathbf{T}_1}(p^{\underline{m}}\mathfrak{R}) \to 0 \tag{1.35}$$

where $\mathscr{U}_{\underline{m}} := j(1 + p^{\underline{m}}\mathbb{Z}_p \otimes \mathcal{O}_{F_+})$ for j from subsection 1.2.3 and $\mathrm{Cl}_{\mathbf{T}_1}(p^{\underline{m}}\mathfrak{R}) := \Gamma_{p^\infty\mathfrak{R}}^{\mathrm{ac}}/\mathscr{U}_{\underline{m}}$. We then define

$$\mathcal{A}_{\underline{m}}(\Gamma_{p^\infty\mathfrak{R}}^{\mathrm{ac}}, L) := \{f \in \mathcal{A}(\Gamma_{p^\infty\mathfrak{R}}^{\mathrm{ac}}, L) : f \text{ is analytic on every translation of } \mathscr{U}_{\underline{m}}\}. \tag{1.36}$$

Each $\mathcal{A}_{\underline{m}}$ is a Banach L-space with a discretely valued norm $||\cdot||_{\underline{m}}$. Furthermore, dualising gives a family of norms on $\mathcal{D}(\Gamma_{p^\infty\mathfrak{R}}^{\mathrm{ac}}, L)$ and makes it into a Fréchet module via

$$||\mu||_{\underline{m}} := \sup_{||f||_{\underline{m}} \leq 1} |\mu(f)|. \tag{1.37}$$

If $\mu \in \mathcal{D}(\Gamma_{p^\infty\mathfrak{R}}^{\mathrm{ac}}, L)$, we then define it to be of growth $\underline{h} := (h_{\mathfrak{p}})_{\mathfrak{p}|p} \in \mathbb{Q}_{\geq 0}^{\mathfrak{p}|p}$ if there exists $C \geq 0$ such that for all $\underline{m} \in \mathbb{Z}_{\geq 1}^{\mathfrak{p}|p}$ we have $||\mu||_{\underline{m}} \leq Cp^{\underline{hm}}$ where $\underline{hm} = (h_{\mathfrak{p}}m_{\mathfrak{p}})_{\mathfrak{p}}$. We let $\mathcal{D}^{\underline{h}}(\Gamma_{p^\infty\mathfrak{R}}^{\mathrm{ac}}, L)$ denote the submodule with growth at most $\underline{h}$.

1.3 The main results

For our main results, we will have three key assumptions. We will consider the regular algebraic automorphic representation π of $\mathbf{G}(\mathbb{A}_{\mathbb{Q}})$ of cohomological weight $\underline{\lambda}_\pi$ under the assumptions

1. The weight $\underline{\lambda}_\pi$ has individual pure weight $\underline{w} = (w_\sigma)_{\sigma \in \Sigma_{F^+}}$, and
2. There exists a Hecke character $\eta : \mathbf{T}_1(F^+) \setminus \mathbf{T}_1(\mathbb{A}_{F^+}) \to \mathbb{C}^\times$ such that
   (a) For the central character ω_π of π, we have $\omega_\pi = \eta^n$, and
   (b) The Hecke character η is admissible of infinity type $\underline{w}$.

When BC_{F/F^+} is cuspidal, these two assumptions are strictly weaker than $L(\mathrm{BC}_{F/F^+}(\pi), \wedge^2 \otimes \tilde{\eta}^{-1}, s)$, the exterior product, having a pole at $s = 1$, this being condition 1 of conjecture 1.1 and equivalent to there existing a $(\tilde{\eta}, \psi)$-Shalika model in the sense of [BDGJW25][Theorem 2.1]. Furthermore, we also assume

3. For a relevant parabolic $\mathbf{P}$ and non-$\mathbf{P}$-critical $\mathbf{P}$-refined $\tilde{\pi}$, we have an open compact subgroup $K_f = \mathrm{Iw}_{\mathbf{P}} \prod_{\mathfrak{q}\nmid p} K_{\mathfrak{q}}$ such that there is an E-model $\tilde{\pi}_{f,E}^{K}$ of $\tilde{\pi}_f^K$ for a sufficiently large number field E which embeds in L.

We will call these the *standard assumptions on* π.

Theorem A. *Suppose we have the standard assumptions,* $v_p(\alpha^{\circ}_{\mathbf{P},p}) \neq \infty$ *and* $\varphi \in \tilde{\pi}_{f,E}^{\mathrm{Iw}_{\mathbf{P}}}$. *Then there exists a distribution*

$$\mu_{\underline{\lambda}_{\tilde{\pi}}}(\varphi) \in \mathcal{D}^{\underline{h}}(\Gamma^{ac}_{p^\infty\mathfrak{R}}, L) \tag{1.38}$$

of growth at most $\underline{h} = (v_p(\alpha^{\circ}_{\mathbf{P},\mathfrak{p}}))_{\mathfrak{p}|p}$ *such that for every anti-cyclotomic Hecke character* $\chi \in \mathrm{Crit}(\underline{\lambda}_\pi)[p^\beta\mathfrak{R}]$ *with any* β, *we have that*

$$\mu_{\underline{\lambda}_{\tilde{\pi}}}(\chi_{[p]}) = \frac{\delta_{\mathbf{B}}(t^{-\beta}_{\mathbf{P},p})}{(\alpha^{\circ}_{\mathbf{P},p})^\beta} \mathscr{P}^{\infty}_{\mathbf{H},\chi,\eta}((ut^{\beta}_{p,\mathbf{P}}\varphi_f) \otimes v^{\vee}_{\underline{\lambda}_\pi,\underline{j}_\chi}). \tag{1.39}$$

where $v_{\underline{\lambda}_\pi,\underline{j}_\chi}$ *is a normalised (using lemma 3.3) unique up to scalar basis for* $V_{(\underline{j}_\chi,-\underline{j}_\chi)} \subseteq V_{\underline{\lambda}}$.

Remark 1.4. *It is expected that the above distributions that satisfy* $h_{\mathfrak{p}} < \max_{\sigma\in\Sigma_{\mathfrak{p}}} \#\,\mathrm{Crit}(\lambda_\sigma)$ *are uniquely determined by the interpolation. This will be explored in forthcoming work of the author.*

From the remaining theorems, we assume K_v are hyper special maximal compact for π_v unramified, S is a finite set containing these v, we have global multiplicity $m(\pi) = 1$ which is true if $\mathrm{BC}_{F/F^+}(\pi)$ is cuspidal and $K_f^{\mathbf{P}} = K^p\,\mathrm{Iw}_{\mathbf{P}}$ and $K = K_f\mathbf{G}(\mathbb{R})$.

Theorem B. *Suppose the assumptions of theorem A are valid. Then for every* $h \geq v_p(\alpha^{\circ}_{\mathbf{P},p})$, *there exists an* h*-adapted neighbourhood* $\Omega \subseteq \mathscr{W}_{K,0}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}(L)$ *of* $\underline{\lambda}_\pi$ *and family* $\tilde{\pi} \in \mathscr{C} \subseteq \mathscr{E}^{\mathbf{P}}$ *such that:*

$$w : \mathscr{C} \cong \Omega \text{ and } w(\tilde{\pi}) = \underline{\lambda}_{\tilde{\pi}} \tag{1.40}$$

and it satisfies the following properties:

1. *We have* $\dim \mathscr{C} = \dim \Omega = \dim \mathscr{W}_{K,0}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}(L)$, *and*
2. *The subset* $\mathscr{C}_{cl}$ *of classical points is Zariski-dense in* $\mathscr{C}$.

Theorem C. *Suppose we have a family* $\mathscr{C}$ *and neighbourhood* Ω *as in theorem B and let* $\mathscr{C}_{scl}$ *be the of classical points which also satisfy the standard assumptions. Then there exists a distribution*

$$\mu_\Omega(\varphi) \in \mathcal{D}(\Gamma^{ac}_{p^\infty\mathfrak{R}}, \mathcal{O}_\Omega) \tag{1.41}$$

such that for every $\omega \in \Omega_{scl}$, *there exist p-adic periods* $c_\omega \in L^\times$ *and* $\varphi_{\omega,f} \in \pi_{\omega,f,E_\omega}^{K_f^{\mathbf{P}}}$ *such that we have:*

$$sp_\omega(\mu_\Omega(\varphi)) = c_\omega \mu_\omega(\varphi_\omega). \tag{1.42}$$

Furthermore, if $\mathscr{C}_{scl}$ *is Zariski-dense in* $\mathscr{C}$, *then* $\mu_\Omega(\varphi)$ *is unique up to units in* $\mathcal{O}_\Omega^\times$.

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2 Classical evaluation maps

In this section, we will seek to repackage the unitary Friedberg–Jaquett integral into a device which takes algebraic input and yields us the period integral values, this being the evaluation map. For the case of this article, we begin with the convenient observation that $\mathbf{G}$ is compact at infinity which allows us to take advantage the algebraic modular forms of [Gro99] and make our calculations relatively explicit.

2.1 The period integral

We will first observe that the period integral can be rewritten. Let π be a RACAR of $\mathbf{G}$, χ an anti-cyclotomic Hecke character, $K_f \subseteq \mathbf{G}(\mathbb{A}_{\mathbb{Q},f})$ an open compact subgroup such that $\chi'_f(\det(h_1^{-1}h_2))$ and $\eta(\det h_2^{-1})$ are trivial on K_f, $K_{f,\mathbf{H}} = \mathbf{H}(\mathbb{A}_{\mathbb{Q},f}) \cap K_f$ and a vector $\varphi \in \pi_\infty \otimes \pi_f^{K_f}$. We also introduce the r representatives h_i of $[\mathbf{H}]/\mathbf{G}(\mathbb{R})K_{f,\mathbf{H}}$. Then we have that

$$\begin{aligned}
\mathscr{P}^{\infty}_{\mathbf{H},\chi,\eta}(\varphi) &= \int_{[\mathbf{H}]} \varphi(h)\eta\left(\frac{1}{\det h_2}\right)\chi'\left(\frac{\det h_2}{\det h_1}\right)dh \\
&= \sum_{i=1}^{r} \int_{\mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{Q})h_i A_{\mathbf{H}}\mathbf{H}(\mathbb{R})K_{f,\mathbf{H}}} \varphi(h)\eta\left(\frac{1}{\det h_2}\right)\chi'\left(\frac{\det h_2}{\det h_1}\right)dh \\
&= \sum_{i=1}^{r} \eta\left(\frac{1}{\det h_{i,2}}\right)\chi'\left(\frac{\det h_{i,2}}{\det h_{i,1}}\right) \\
&\cdot \int_{\mathrm{Stab}_{\mathbf{H}(\mathbb{Q})}(h_i K_{f,\mathbf{H}})\backslash\mathbf{H}(\mathbb{R})} \varphi(h_i h_\infty)\eta\left(\frac{1}{\det h_{\infty,2}}\right)\chi'\left(\frac{\det h_{\infty,2}}{\det h_{\infty,1}}\right)dh \\
&= \mathrm{Vol}(K_{f,\mathbf{H}}) \sum_{i=1}^{r} \frac{1}{|\mathbf{H}(\mathbb{Q}) \cap h_i K_{f,\mathbf{H}} h_i^{-1}|}\eta\left(\frac{1}{\det h_{i,2}}\right)\chi'\left(\frac{\det h_{i,2}}{\det h_{i.1}}\right) \\
&\cdot \int_{\mathbf{H}(\mathbb{R})} \varphi(h_i h_\infty)\eta\left(\frac{1}{\det h_{\infty,2}}\right)\chi'\left(\frac{\det h_{\infty,2}}{\det h_{\infty,1}}\right)dh_\infty \\
&= \mathrm{Vol}(K_{f,\mathbf{H}}) \sum_{i=1}^{r} \frac{\eta(\det h_{i,2}^{-1})\chi'\left(\det(h_{i,1}^{-1})\det(h_{i,2})\right)}{|\mathbf{H}(\mathbb{Q}) \cap h_i K_{f,\mathbf{H}} h_i^{-1}|}(e_{(\chi'_\infty,(\chi'_\infty)^{-1}\otimes\eta_\infty)}\varphi)(h_i)
\end{aligned}$$

where $A_{\mathbf{H}} = \{1\}$ is the maximal $\mathbb{Q}$-split subtorus of $\mathbf{H}$, dh is the Tamagawa measure such that $dh = dh_f dh_\infty$ is fixed by $\mathrm{Vol}(\mathbf{H}(\mathbb{R})) = 1$, note that this also fixes dh_f and makes Vol well-defined, $e_{(\chi'_\infty,(\chi'_\infty)^{-1}\otimes\eta_\infty)}$ projects $\pi_\infty|_{\mathbf{H}(\mathbb{R})}$ into the $(\chi'_\infty,(\chi'_\infty)^{-1}\otimes\eta_\infty)$-isotopic $\mathbf{H}(\mathbb{R})$-space $V_{(\underline{j}_\chi,-\underline{j}_\chi-\underline{w}_\pi)}$ and we use $\mathrm{Stab}_{\mathbf{H}(\mathbb{Q})}(h_i K_{f,\mathbf{H}}) = \mathbf{H}(\mathbb{Q}) \cap h_i K_{f,\mathbf{H}} h_i^{-1}$. Recalling from equation 1.27 that $\pi_\infty \cong V_{\underline{\lambda}_\pi}(\mathbb{C})$, we now find a suitable $\varphi_\infty \in \pi_\infty$. To do this, we will use the dual space $V_{(\underline{j}_\chi,-\underline{j}_\chi-\underline{w}_\pi)} \cong V^\vee_{(-\underline{j}_\chi,+\underline{j}_\chi+\underline{w}_\pi)}$. First, we have from [SDW24, lemma 5.2.] combined with Weyl's complete reducibility theorem

$$\underline{j} \in \mathrm{Crit}(\underline{\lambda}) \Leftrightarrow V_{\underline{\lambda}}|_{(\mathbf{GL}_n\times\mathbf{GL}_n)} = W \oplus V_{(-\underline{j},\underline{w}+\underline{j})} \tag{2.1}$$

where W contains no copies of $V_{(-\underline{j},\underline{w}+\underline{j})}$. We let $v_{\underline{\lambda},\underline{j}}$ be the basis of $V_{(-\underline{j},\underline{w}+\underline{j})} \subseteq V_{\underline{\lambda}}|_{(\mathbf{GL}_n\times\mathbf{GL}_n)}$ and define $v^\vee_{\underline{\lambda},\underline{j}} \in V^\vee_{(-\underline{j},\underline{w}+\underline{j})}$ as the element which satisfies $v^\vee_{\underline{\lambda},\underline{j}}(v_{\underline{\lambda},\underline{j}}) = 1$. This permits us to describe a suitable test vectors at infinity

$$v_{\underline{\lambda},\infty} := \sum_{\underline{j}\in\mathrm{Crit}(\underline{\lambda})} v_{\underline{\lambda},\underline{j}}. \tag{2.2}$$

In particular, we let $\kappa^\infty_{\underline{\lambda},\underline{j}}$ be the continuous dual of the inclusion $V_{(-\underline{j},\underline{j}+\underline{w})} \subseteq V_{\underline{\lambda}}|_{(\mathbf{GL}_n\times\mathbf{GL}_n)}$ and observe that $e_{(\chi'_\infty,(\chi'_\infty)^{-1}\otimes\eta_\infty)}v^\vee_{\underline{\lambda}_\pi,\infty} = \kappa^\infty_{\underline{\lambda}_\pi,\underline{j}_\chi}(v^\vee_{\underline{\lambda}_\pi,\infty}) = v^\vee_{\underline{\lambda}_\pi,\underline{j}_\chi} \in V^\vee_{(-\underline{j}_\chi,+\underline{j}_\chi+\underline{w}_\pi)} \cong V_{(\underline{j}_\chi,-\underline{j}_\chi-\underline{w}_\pi)}$.

2.2 Algebraic modular forms

We now seek to repackage the automorphic representations in terms of algebraic modular forms. For open compact subgroups $K_f \subseteq \mathbf{G}(\mathbb{A}_{\mathbb{Q},f})$ and V a $\mathbb{Q}$-vector space and $\mathbf{G}(\mathbb{Q})$-representation, we define algebraic modular forms as

$$M(V, K_f) := \{f : \mathbf{G}(\mathbb{A}_{\mathbb{Q}})/(\mathbf{G}(\mathbb{R}) \times K_f) \to V \text{ such that } f(\gamma g) = \gamma f(g) \text{ for } \gamma \in \mathbf{G}(\mathbb{Q})\}. \tag{2.3}$$

As described in [Gro99, section 8], we have related archimedian and p-adic realisations. Suppose L is an archimedian or p-adic field and V a L-vector space. Depending on L, we then respectively have

$$M_\infty(V, K_f) := \{\mathcal{F} : \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{Q}})/K_f \to V \text{ such that } \mathcal{F}(\gamma g_\infty) = g_\infty^{-1}\mathcal{F}(g) \text{ for } g_\infty \in \mathbf{G}(\mathbb{R})\}, \tag{2.4}$$

$$M_p(V, K_f) := \{\mathcal{F} : \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{Q}})/(\mathbf{G}(\mathbb{R}) \times K_f^p) \to V \text{ s.t. } \mathcal{F}(gg_p) = g_p^{-1}\mathcal{F}(g) \text{ for } g_p \in K_p\}, \tag{2.5}$$

where the last assumes $K_f = K_p K^p$ with $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ an open compact subgroup. Furthermore, when V is a rational vector space, these are naturally connected to base change via [Gro99, Proposition 8.3 and 8.6]:

$$M(V, K_f) \otimes L \cong M_\infty(V \otimes L, K_f); f \otimes l \mapsto l \cdot g_\infty^{-1} f(g), \tag{2.6}$$

$$M(V, K_f) \otimes L \cong M_p(V \otimes L, K_f); f \otimes l \mapsto l \cdot g_p^{-1} f(g). \tag{2.7}$$

We now check that we can describe automorphic forms via algebraic modular forms. Given an automorphic representation $\pi = \pi_\infty \otimes_{v\nmid\infty} \pi_v$, we have a pairing $\pi_\infty^\vee \times \pi_\infty \to \mathbb{C}$, this being the same as the invariant inner product $\langle\cdot,\cdot\rangle$ of [Gro99, Page 76]. Recalling that $\pi_\infty \cong V_{\underline{\lambda}_\pi}(\mathbb{C})$, this gives for all $\mathcal{F} \in M_\infty(V_{\underline{\lambda}_\pi}^\vee(\mathbb{C}), K_f)$ a map

$$\ell_{\mathcal{F}} : V_{\underline{\lambda}_\pi}^\vee(\mathbb{C}) \to L^2([\mathbf{G}]); v \mapsto (g \mapsto \langle \mathcal{F}(g), v\rangle). \tag{2.8}$$

Moreover, by [Gro99, Proposition 8.5], this map yields an isomorphism

$$M_\infty(V_{\underline{\lambda}_\pi}^\vee(\mathbb{C}), K_f) \cong \bigoplus_{\pi' \subseteq L_2([\mathbf{G}])} \mathrm{Hom}_{\mathbf{G}(\mathbb{R})}(V_{\underline{\lambda}_\pi}^\vee(\mathbb{C}), \pi'_\infty \otimes (\pi'_f)^{K_f}). \tag{2.9}$$

where we allow two $\pi', \pi'' \subseteq L_2([\mathbf{G}])$ to be isomorphic but not equal. Furthermore, using that $(\pi'_f)^{K_f}$ is finite dimensional, we can cut out the π isotopic part.

$$M_\infty(V_{\underline{\lambda}_\pi}^\vee(\mathbb{C}), K_f)[\pi] \cong \mathrm{Hom}_{\mathbf{G}(\mathbb{R})}(V_{\underline{\lambda}_\pi}^\vee(\mathbb{C}), \pi_\infty) \otimes \pi_f^{K_f} \otimes \mathrm{Hom}_{\mathbf{G}(\mathbb{A}_{\mathbb{Q}})}(\pi, L([\mathbf{G}])). \tag{2.10}$$

where $\mathrm{Hom}_{\mathbf{G}(\mathbb{A}_{\mathbb{Q}})}(\pi, L([\mathbf{G}]))$ accounts for potential multiplicities of π. Using Schur's lemma, we may rewrite this as

$$M_\infty(V_{\underline{\lambda}_\pi}^\vee(\mathbb{C}), K_f)[\pi] \cong \pi_f^{K_f} \otimes \mathrm{Hom}_{\mathbf{G}(\mathbb{A}_{\mathbb{Q}})}(\pi, L([\mathbf{G}])). \tag{2.11}$$

Moreover, given a number field E, this isomorphism descends to the E-structure. To do this, we critically observe that we can rewrite $\mathrm{Hom}_{\mathbf{G}(\mathbb{A}_{\mathbb{Q}})}(\pi, L([\mathbf{G}])) \cong \mathrm{Hom}_{\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})}(\pi_f, \varinjlim_{K_f} M_\infty(V_{\underline{\lambda}_\pi}^\vee(\mathbb{C}), K_f)[\pi])$, and the last has a natural E-structure.

Lemma 2.1. *Suppose E is a sufficiently large number field, π_f has an E-model $\pi_{f,E}$ and $f(g) = g_\infty^{-1}\mathcal{F}(g) \in M(V_{\underline{\lambda}_\pi}(E)^\vee, K_f)$. Then*

$$\ell_{F,E} : V_{\underline{\lambda}_\pi}^\vee(E) \to \pi_{f,E}^{K_f} \otimes \mathrm{Hom}_{\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})}(\pi_{E,f}, \varinjlim_{K_f} M(V_{\underline{\lambda}_\pi}^\vee(E), K_f)[\pi]). \tag{2.12}$$

induces the following commutative diagram

$$\begin{array}{ccc} M_\infty(V_{\underline{\lambda}}(\mathbb{C})^\vee, K_f)[\pi] & \longrightarrow & \pi_f^{K_f} \otimes \mathrm{Hom}_{\mathbf{G}(\mathbb{A}_\mathbb{Q})}(\pi, L([\mathbf{G}])) \\ \uparrow & & \uparrow \\ M(V_{\underline{\lambda}}^\vee(E), K_f)[\pi_E] & \longrightarrow & \pi_{f,E}^{K_f} \otimes \mathrm{Hom}_{\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})}(\pi_{E,f}, \varinjlim_{K_f} M(V_{\underline{\lambda}}^\vee(E), K_f)[\pi]) \end{array}$$

where the horizontal maps are isomorphisms.

Proof. This follows from the base change of $\mathbb{C}/E$ being flat and equation 2.9. □

Consequently, we can find suitable corresponding algebraic modular forms.

Lemma 2.2. *Let $\varphi_f \in \pi_f$ with $K_f \subseteq \mathbf{G}(\mathbb{A}_{\mathbb{Q},f})$ an open compact subgroup such that $\varphi_f \in \pi_f^{K_f}$. For $\mathcal{F}(g_f g_\infty) := \varphi_f(g_f) \cdot (g_\infty^{-1} v_{\underline{\lambda}_\pi,\infty}^\vee) \in M_\infty(V_{\underline{\lambda}_\pi}^\vee(\mathbb{C}), K_f)[\pi]$, we have:*

1. *$\kappa_{\underline{\lambda}_\pi,\underline{j}} \circ \mathcal{F}(g) \in L_2([\mathbf{G}])$, and*
2. *$\kappa_{\underline{\lambda}_\pi,\underline{j}} \circ \mathcal{F}(g) = (\varphi_f \otimes v_{\underline{\lambda}_\pi,\underline{j}}^\vee)(g)$.*
3. *If $\varphi_f \in \pi_{E,f}$, then $f(g) := g_\infty \mathcal{F}(g) = \varphi_f(g_f) v_{\underline{\lambda}_\pi^\vee,\infty} \in M(V_{\underline{\lambda}_\pi}^\vee(E), K_f)$.*

Proof. From directly checking, it clear that $\mathcal{F} \in M_\infty(V_{\underline{\lambda}_\pi}^\vee(\mathbb{C}), K_f)$, and we further have via equation 2.10 and equation 2.8 that it belongs to the π-isotypic component. For part a), we then note that the map $\kappa_{\underline{\lambda}_\pi,\underline{j}}$ is another realisation of the linear map in $\ell_\mathcal{F}$, that being:

$$\kappa_{\underline{\lambda}_\pi,\underline{j}}(\mathcal{F}(g)) = \ell_\mathcal{F}(v_{\underline{\lambda}_\pi,\underline{j}}^\vee)(g) := \langle \mathcal{F}(g), v_{\underline{\lambda}_\pi,\underline{j}}^\vee \rangle \in L_2([\mathbf{G}]). \tag{2.13}$$

For part b), we then have:

$$\kappa_{\underline{\lambda}_\pi,\underline{j}}(\mathcal{F}(g)) = \kappa_{\underline{\lambda}_\pi,\underline{j}}(\varphi_f(g_f) \cdot g_\infty^{-1} v_{\underline{\lambda}_\pi,\infty}^\vee) = \varphi_f(g_f) \cdot \kappa_{\underline{\lambda}_\pi,\underline{j}}(v_{\underline{\lambda}_\pi,\infty}^\vee) = \varphi_f(g_f) \cdot v_{\underline{\lambda}_\pi,\underline{j}}^\vee(g_\infty). \tag{2.14}$$

Lastly, part c) follows directly from lemma 2.1 □

2.3 The evaluation maps

We can now use the relationship between automorphic representations and algebraic modular forms $\mathcal{F} \in M_\infty(V_{\underline{\lambda}}^\vee(\mathbb{C}), K_f)$, Hecke characters η and anti-cyclotomic $\chi \in \mathrm{Crit}(\underline{\lambda})$ of infinity type $-\underline{j}_\chi$ to define an evaluation map $\mathrm{Ev}_{\underline{\lambda},\chi}^\infty : M_\infty(V_{\underline{\lambda}}^\vee, K_f) \to \mathbb{C}$

$$\mathrm{Ev}_{\underline{\lambda},\chi}^\infty(\mathcal{F}) := \mathscr{P}_{\mathbf{H},\chi,\eta}^\infty(\kappa_{\underline{\lambda},\underline{j}_\chi}^\infty \circ \mathcal{F}) = \mathrm{Vol}(K_{f,\mathbf{H}}) \sum_{i=1}^r \frac{\kappa_{\underline{\lambda},\underline{j}_\chi}^\infty \circ \mathcal{F}(h_i) \eta\left(\det h_2^{-1}\right) \chi_f'\left(\det(h_{i,1}^{-1}) \det(h_{i,2})\right)}{|\mathbf{H}(\mathbb{Q}) \cap h_i K_{f,\mathbf{H}} h_i^{-1}|} \tag{2.15}$$

where we have used that $e_{(\chi_\infty',(\chi_\infty')^{-1}\otimes\eta_\infty)}(\kappa_{\underline{\lambda},\underline{j}_\chi} \circ \mathcal{F}) = \kappa_{\underline{\lambda},\underline{j}_\chi}^\infty \circ \mathcal{F}$. We can now use this finite sum description to give a p-adic analogue. Analogously to $\kappa_{\underline{\lambda},\underline{j}}^\infty$, $L/\mathbb{Q}_p$ we define $\kappa_{\underline{\lambda},\underline{j}}^p$ to be the continuous dual of the inclusion $V_{(-\underline{j},\underline{w}+\underline{j})}(L) \subseteq V_{\underline{\lambda}}(L)$.

Proposition 2.3. *Let $\underline{\lambda} \in X_0^*(\mathbf{T})$, E a number field, $L/\mathbb{Q}_p$ a p-adic field containing E, $\chi \in \mathrm{Crit}(\underline{\lambda})$ of infinity type $\underline{j}_\chi$ and $K_f \subseteq \mathbf{G}(\mathbb{A}_{f,\mathbb{Q}})$ an open compact subgroup such that $(h_1, h_2) \mapsto \chi_f'(\det(h_1^{-1} h_2))$ and $h_2 \mapsto \eta(\det h_2^{-1})$ is trivial on $K_{f,\mathbf{H}}$. Then we have a well-defined map*

$$\mathrm{Ev}_{\underline{\lambda},\chi}^p : M_p(V_{\underline{\lambda}}^\vee(L), K_f) \to L; \mathcal{F} \mapsto \mathrm{Vol}(K_{f,\mathbf{H}}) \sum_{i=1}^r \frac{\kappa_{\underline{\lambda},\underline{j}_\chi}^p \circ \mathcal{F}(h_i) \eta\left(\det h_2^{-1}\right) \chi_f'\left(\det(h_{i,1}^{-1}) \det(h_{i,2})\right)}{|\mathbf{H}(\mathbb{Q}) \cap h_i K_{f,\mathbf{H}} h_i^{-1}|} \tag{2.16}$$

and we have for $f \in M(V_{\underline{\lambda}}^{\vee}(E), K_f)$ that:

$$\mathrm{Ev}_{\underline{\lambda},\chi}^{p}(g_p^{-1} f(g)) = \mathrm{Ev}_{\underline{\lambda},\chi}^{\infty}(g_\infty^{-1} f) \in E(\chi) \tag{2.17}$$

where $\mathbb{Q}(\chi)$ is the smallest number field containing the image of the Hecke character χ.

Note that E is an avatar for the extension of $\mathbb{Q}$ containing the Hecke eigenvalues with respect to π. Furthermore, we note that the property of the finite parts of χ and η being trivial on K_f is critical. If we define the algebraic infinity part of any Hecke character χ' as $\omega^{\underline{j}}(z) := \prod_{\sigma \in \Sigma_{F^+}} \sigma(z)^{j_\sigma}$, then for $k_i = k_{p,i} k_{\mathfrak{R},i} k_i'$ where $k_{p,i}$ is the p-part, $k_{\mathfrak{R},i}$ is the $\mathfrak{R}$ part for $p \nmid \mathfrak{R}$ we have

$$\chi_{[p]}(\det(k_{p,2} k_{\mathfrak{R},2} k_{p,1}^{-1} k_{\mathfrak{R},1}^{-1})) = \omega_p^{\underline{j}}(\det(k_{p,2} k_{p,1}^{-1})) \tag{2.18}$$

where $\omega_p^{\underline{j}}$ is the base change of $\omega^{\underline{j}}$ to $\mathbb{Z}_p$.

Proof. We begin by checking that this is well-defined. To do this, we have to check that the evaluation map is independent of the representatives as the maps are no longer $K_{f,\mathbf{H}}$-invariant, but instead $K_{p,\mathbf{H}} \subseteq \mathbf{G}(\mathbb{Q}_p)$-equivariant. If we let $k_p \in K_{p,\mathbf{H}}$, then it suffices to show that $h_i k_p$ yields the same term as h_i.

$$\begin{aligned}
&(\kappa_{\underline{\lambda},\underline{j},\underline{\chi}}^{p} \circ \mathcal{F})(h_i k_p) \eta_{[p]}(\det h_{i,2}^{-1} k_{i,2}^{-1}) \chi'_{[p]}(\det((h_{i,1} k_{p,1})^{-1} h_{i,2} k_{p,2})) \\
&= \kappa_{\underline{\lambda},\underline{j},\underline{\chi}}^{p}(k_p^{-1} \mathcal{F}(h_i)) \omega_p^{\underline{w}}\left(\det k_{i,2}^{-1}\right) \eta_{[p]}(\det h_{i,2}^{-1}) \omega_p^{-\underline{j}_\chi} \left(\frac{\det(k_{p,2})}{\det(k_{p,1})}\right) \chi'_{[p]}(\det(h_{i,1}^{-1} h_{i,2})) \\
&= \frac{\omega_p^{\underline{j}_\chi}(\det(k_{p,1}))}{\omega_p^{\underline{j}_\chi + \underline{w}}(\det(k_{p,2}))} (k_p^{-1} * \kappa_{\underline{\lambda},\underline{j},\underline{\chi}}^{p}(\mathcal{F}(h_i))) \eta_{[p]}(\det h_{i,2}^{-1}) \chi'_{[p]}(\det(h_{i,1}^{-1} h_{i,2})) \\
&= \kappa_{\underline{\lambda},\underline{j},\underline{\chi}}^{p}(\mathcal{F}(h_i)) \eta_{[p]}(\det h_{i,2}^{-1}) \chi'_{[p]}(\det(h_{i,1}^{-1} h_{i,2})).
\end{aligned}$$

In the first equality, we use equation 2.18, in the second equality we refactor the $\omega_p^{\bullet}$ functions and use the $\mathbf{H}(\mathbb{Q}_p)$-equivariance of $\kappa_{\underline{\lambda},\underline{j},\underline{\chi}}^{p}$, and in the last equality we observe the k_p^{-1}-action cancels out the $\omega_p^{\bullet}$ factors. It then remains to prove the second equality. We recall $f \in M(V_{\underline{\lambda}}^{\vee}(E), K_f)$ embeds as $M_\infty(V_{\underline{\lambda}}^{\vee}(\mathbb{C}), K_f)$ and $M_p(V_{\underline{\lambda}}^{\vee}(L), K_f)$ via $g_\infty^{-1} f(g)$ and $g_p^{-1} f(g)$. Thus it follows

$$\begin{aligned}
&\mathrm{Ev}_{\underline{\lambda},\chi}^{p}(g_p^{-1} f(g)) \\
&= \mathrm{Vol}(K_{f,\mathbf{H}}) \sum_{i=1}^{r} \frac{\kappa_{\underline{\lambda},\underline{j},\underline{\chi}}^{p}(h_p^{-1} f(h_i)) \eta_{[p]}(\det h_{i,2}^{-1}) \chi'_{[p]}\left(\det(h_{i,1}^{-1} h_{i,2})\right)}{|\mathbf{H}(\mathbb{Q}) \cap h_i K_{f,\mathbf{H}} h_i^{-1}|} \\
&= \mathrm{Vol}(K_{f,\mathbf{H}}) \sum_{i=1}^{r} \frac{h_p^{-1} * (\kappa_{\underline{\lambda},\underline{j},\underline{\chi}}^{p} \circ f(h_i)) \omega_p^{\underline{w}}(\det h_{i,2}^{-1}) \omega_p^{-\underline{j}}(\det(h_{1,p}^{-1} h_{2,p})) \eta_f(\det h_{i,2}^{-1}) \chi'_f\left(\det(h_{i,1}^{-1} h_{i,2})\right)}{|\mathbf{H}(\mathbb{Q}) \cap h_i K_{f,\mathbf{H}} h_i^{-1}|} \\
&= \mathrm{Vol}(K_{f,\mathbf{H}}) \sum_{i=1}^{r} \frac{h_\infty^{-1} * (\kappa_{\underline{\lambda},\underline{j},\underline{\chi}}^{\infty} \circ f(h_i)) \omega_\infty^{\underline{w}}(\det h_{i,2}^{-1}) \omega_\infty^{-\underline{j}}(\det(h_{1,p}^{-1} h_{2,p})) \eta_f(\det h_{i,2}^{-1}) \chi'_f\left(\det(h_{i,1}^{-1} h_{i,2})\right)}{|\mathbf{H}(\mathbb{Q}) \cap h_i K_{f,\mathbf{H}} h_i^{-1}|} \\
&= \mathrm{Vol}(K_{f,\mathbf{H}}) \sum_{i=1}^{r} \frac{\kappa_{\underline{\lambda},\underline{j},\underline{\chi}}^{\infty}(h_{i,\infty}^{-1} f(h_i)) \eta(\det h_{i,2}^{-1}) \chi'\left(\det(h_{i,1}^{-1} h_{i,2})\right)}{|\mathbf{H}(\mathbb{Q}) \cap h_i K_{f,\mathbf{H}} h_i^{-1}|} = \mathrm{Ev}_{\underline{\lambda},\chi}^{\infty}(g_\infty^{-1} f(g)).
\end{aligned}$$

where the first equality unpacks the definition, the expands out the $\omega_p^{\bullet}$ part of the p-adic Hecke characters and uses the $\mathbf{H}(\mathbb{Q}_p)$-equivariance of $\kappa_{\underline{\lambda},\underline{j},\underline{\chi}}^{p}$, the third uses that $h_{i,p} = h_{i,\infty}$ through the embedding of E in $\mathbb{C}$ and L, the forth equality uses the $\mathbf{H}(\mathbb{R})$-equivariance of $\kappa_{\underline{\lambda},\underline{j},\underline{\chi}}^{\infty}$ and combines the infinity part and finite part of the Hecke characters and the last equality is the definition. It remains to conclude that they are contained in the claimed number field. This follows from noting that $\mathrm{Vol}(K_{f,\mathbf{H}}) \in \mathbb{Q}$, $|\mathbf{H}(\mathbb{Q}) \cap h_i K_{f,\mathbf{H}} h_i^{-1}| \in \mathbb{Q}$, $\mathrm{Im}(\kappa_{\underline{\lambda},\chi} \circ f) \in E$ and $\mathrm{Im}\, \chi' \in \mathbb{Q}(\chi)$. □

3 Branching laws

To lift the evaluation map, we will not have to rewrite the previous evaluation map in terms of distributions. Hence, the goal of this chapter will be to remove the main obstruction by lifting the map $\kappa^p_{\underline{\lambda},\underline{j}}$. This setup takes inspiration from [BDGJW25], but with a purely representation theoretic perspective.

3.1 Parametrisation of the classical branching law

We recall from subsection 1.2.8 that it suffices to only consider the R-valued representations of $\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{GL}_{2n}$ such that $(\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{GL}_{2n})_R \cong \mathbf{G}_R$, and we will be working with $\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{GL}_{2n}$ with $R = F^+ \otimes \mathbb{Q}_p$ and $R = L$. Recalling the fixed embeddings $\sigma_{\mathfrak{p}} : F^+_{\mathfrak{p}} \hookrightarrow L$, we then have a linear map $\iota = \prod_{\mathfrak{p}|p} \sigma_{\mathfrak{p}} : F^+ \otimes \mathbb{Q}_p \to L$. Hence, there is a correspondence between $F^+ \otimes \mathbb{Q}_p$-valued and L-valued representations of dominant weight $\underline{\lambda}$:

$$V_{\underline{\lambda}}(F^+ \otimes \mathbb{Q}_p) \otimes_\iota L \cong V_{\underline{\lambda}}(L) \text{ where we identify } \lambda_{\mathfrak{p}} = (\lambda_\sigma)_{\sigma \in \Sigma_{F_{\mathfrak{p}}}}. \tag{3.1}$$

Hence, we have an injective maps which we identify as an inclusions:

$$V_{\underline{\lambda}}(F^+ \otimes \mathbb{Q}_p) \subseteq V_{\underline{\lambda}}(L) \text{ and } V_{\lambda_{\mathfrak{p}}}(F^+_{\mathfrak{p}}) \subseteq \otimes_{\sigma \in \Sigma_{F^+_{\mathfrak{p}}}} V_{\lambda_\sigma}. \tag{3.2}$$

Similarly, we identify a natural inclusion of $X^*_0(\mathbf{T})_\sigma \subseteq X^*_0(\mathbf{T})_{\mathfrak{p}} := \prod_{\tau \in \Sigma_{F^+_{\mathfrak{p}}}} X^*_0(\mathbf{T})_\tau$. We will also identify a correspondence between via the previously fixed embeddings $\sigma_{\mathfrak{p}} : F^+_{\mathfrak{p}} \hookrightarrow L$:

$$\tau \in G_{\mathfrak{p}} := \mathrm{Gal}(F^+_{\mathfrak{p}}/\mathbb{Q}_p) \leftrightarrow \sigma_{\mathfrak{p}} \circ \tau \in \Sigma_{F^+_{\mathfrak{p}}} \subseteq \Sigma_{F^+}. \tag{3.3}$$

We will now seek to parametrise the previous branching law explicitly.

Proposition 3.1. *Let $\sigma \in \Sigma_{F^+}$. The following pure weights*

$$\begin{aligned}
&\alpha_{\sigma,0} = (1, 1, ..., 1)\\
&\alpha_{\sigma,i} = (\underbrace{1, .., 1}_{i \text{ entries}}, 0, .., 0, \underbrace{-1, .. - 1}_{i \text{ entries}}) \quad 1 \le i \le n-1 \quad \textit{(exclude when } n = 1\textit{)}\\
&\alpha_{\sigma,n} = (\underbrace{1, ..., 1}_{n \text{ entries}}, 0, ..., 0)
\end{aligned}$$

are linearly independent and satisfy the following properties:

1. *We have $X^*_0(\mathbf{T})_\sigma = \mathbb{Z}\alpha_{\sigma,0} \oplus \left(\bigoplus_{1 \le i \le n} \mathbb{Z}_{\ge 0}\alpha_{\sigma,i}\right)$,*
2. *Via the natural inclusion $X^*_0(\mathbf{T})_\sigma \subseteq X^*_0(\mathbf{T})_{\mathfrak{p}}$, there exist $(\mathbf{GL}_n \times \mathbf{GL}_n)_{\mathfrak{p}_\sigma}$-eigenvectors:*
   (a) *$v_{\sigma,0} \in V_{\alpha_{\sigma,0}}$ such that for $h_{\mathfrak{p}_\sigma} \in (\mathbf{GL}_n \times \mathbf{GL}_n)_{\mathfrak{p}_\sigma}$ we have $h_{\mathfrak{p}_\sigma} \cdot v_{\sigma,0} = \sigma(\det(h_{\mathfrak{p}_\sigma}))v_{\sigma,0}$,*
   (b) *$v_{\sigma,i} \in V_{\alpha_{\sigma,i}}$ for $1 \le i \le n-1$ are $(\mathbf{GL}_n \times \mathbf{GL}_n)_{\mathfrak{p}_\sigma}$-invariant, and*
   (c) *$v_{\sigma,n,j} \in V_{\alpha_{\sigma,n}}$ for $j \in \{1,2\}$ such that for $h_{\mathfrak{p}_\sigma} = (h_{\mathfrak{p},1}, h_{\mathfrak{p},2}) \in (\mathbf{GL}_n \times \mathbf{GL}_n)_{\mathfrak{p}_\sigma}$ we have $h_{\mathfrak{p}_\sigma} \cdot v_{\sigma,n,j} = \sigma(\det(h_{\mathfrak{p},j}))v_{\sigma,n,j}$.*
3. *For all $\tau \in \Sigma_{F^+_{\mathfrak{p}}}$ and $j \in \{1,2\}$, we have:*

$$\tau(v_{\sigma,n,j}) = v_{\sigma\tau,n,j}.$$

The proof of proposition 3.1 will follow the techniques of subsection 11.1 of [BDGJW25].

Proof of proposition 3.1. For the entire proof, we will fix an arbitrary $\sigma \in \Sigma_{F^+}$.

1. For the first part of the proof, we immediately get the result by adding together the components:

$$\lambda_\sigma = \lambda_{\sigma,n+1}\alpha_{\sigma,0} + \sum_{i=0}^{n}(\lambda_{\sigma,i} - \lambda_{\sigma,i+1})\alpha_{\sigma,i}.$$

2. Similar to equation 2.1, we have:

$$-\lambda_{\sigma,n} \leq j_\sigma \leq -\lambda_{\sigma,n+1} \Leftrightarrow V_{\lambda_\sigma}|_{(\mathbf{GL}_n\times\mathbf{GL}_n)_{\mathfrak{p}_\sigma}} = W \oplus V_{(-j_\sigma, w_\sigma+j_\sigma)_\sigma} \tag{3.4}$$

where W contains no copies of $V_{(-j_\sigma,w_\sigma+j_\sigma)_\sigma}$. We can then apply this on each of the weights defined above to get the $(\mathbf{GL}_n \times \mathbf{GL}_n)_{\mathfrak{p}_\sigma}$-subrepresentations $V_{\sigma,(-j_\sigma,w_\sigma+j_\sigma)_\sigma}$. By definition, they are one dimensional with explicit $(\mathbf{GL}_n \times \mathbf{GL}_n)_{\mathfrak{p}_\sigma}$-actions in terms of eigenvalues.

   (a) For $\alpha_{\sigma,0}$, we find $V_{\alpha_{\sigma,0}}$ is the one dimensional representation $g \mapsto \sigma(\det(g))$,

   (b) For $\alpha_{\sigma,i}$ with $1 \leq i \leq n-1$, we have $0 \leq j_\sigma \leq 0$. Thus equation 3.4 yields $V_{(0,0)_\sigma} \subseteq V_{\alpha_{\sigma,i}}$. Picking a basis element for $V_{(0,0)_\sigma}$ yields the desired $(\mathbf{GL}_n \times \mathbf{GL}_n)_{\mathfrak{p}_\sigma}$-eigenvector.

   (c) For $\alpha_{\sigma,n}$, we have $-1 \leq j_\sigma \leq 0$. Thus equation 3.4 yields $V_{(0,1)_\sigma} \oplus V_{(1,0)_\sigma} \subseteq V_{\alpha_{\sigma,n}}$. Picking basis vectors $v_{\sigma,n,1} \in V_{\sigma,(1,0)_\sigma}$ and $v_{\sigma,n,2} \in V_{(0,1)_\sigma}$ yields the desired $(\mathbf{GL}_n \times \mathbf{GL}_n)_{\mathfrak{p}_\sigma}$-eigenvectors.

3. First, we observe that for all $\tau \in \Sigma_{F^+_{\mathfrak{p}_\sigma}}$ we have a commutative diagram of isomorphisms:

$$\begin{array}{ccc} V_{\lambda_\sigma} & \xrightarrow{\tau} & V_{\lambda_{\tau\sigma}} \\ \downarrow{\scriptstyle L_g} & & \downarrow{\scriptstyle L_{\tau(g)}} \\ V_{\lambda_\sigma} & \xrightarrow{\tau} & V_{\lambda_{\tau\sigma}} \end{array}$$

where $L_g(v)$ is the left action map of $g \in \mathbf{GL}_{2n,\mathfrak{p}_\sigma}$ and $\tau(g) \in \mathbf{GL}_{2n,\mathfrak{p}_\sigma}$. It then remains to show that this descends to the 1-dimensional $(\mathbf{GL}_n \times \mathbf{GL}_n)_{\mathfrak{p}_\sigma}$-eigenvectors. To do this, we observe that $L_{(h_{\mathfrak{p}_\sigma,1},h_{\mathfrak{p}_\sigma,2})}(v_{\sigma,n,j}) = \sigma(\det(h_{\mathfrak{p}_\sigma,j}))v_{\sigma,n,j}$ which implies

$$L_{\tau(h_{\mathfrak{p}_\sigma,1},h_{\mathfrak{p}_\sigma,2})}(\tau(v_{\sigma,n,j})) = \sigma(\det(\tau(h_{\mathfrak{p}_\sigma,j})))\tau(v_{\sigma,n,j}) = \sigma\tau(\det(h_{\mathfrak{p}_\sigma,j}))\tau(v_{\sigma,n,j}).$$

We then recall that the branching law implies that this action characterises the vector uniquely up to scalar meaning that we can rescale the eigenvectors such that $\tau(v_{\sigma,n,j}) = v_{\sigma\tau,n,j}$ as required.

□

We then use the above pieces to assemble the desired branching laws. Given dominant weights λ and τ, we note there is an inclusion of highest weight representations $V_{\lambda+\tau} \subseteq V_\lambda \otimes V_\tau$, and we will denote that natural left inverse map as r.

Lemma 3.2. *We have the following $F^+ \otimes \mathbb{Q}_p$-valued representations:*

1. *Let $\lambda_\sigma \in X_0^*(\mathbf{T})_\sigma$, $-\lambda_{\sigma,n} \leq j_\sigma \leq -\lambda_{\sigma,n+1}$ and r the natural projection. We then have:*

$$v_{\lambda_\sigma,j_\sigma} := r\left(v_{\sigma,0}^{\otimes\lambda_{\sigma,n+1}} \otimes \left(\bigotimes_{1\leq i\leq n-1} v_{\sigma,i}^{\otimes(\lambda_{\sigma,i}-\lambda_{\sigma,i+1})}\right) \otimes v_{\sigma,n,1}^{\otimes(-\lambda_{\sigma,n+1}-j_\sigma)} \otimes v_{\sigma,n,2}^{\otimes(\lambda_{\sigma,n}+j_\sigma)}\right) \in V_{\lambda_\sigma},$$

such that $F^+_{\mathfrak{p}_\sigma} v_{\lambda_\sigma,j_\sigma} = V_{(-j_\sigma,w_\sigma+j_\sigma)_\sigma} \subseteq V_{\lambda_\sigma}|_{(\mathbf{GL}_n\times\mathbf{GL}_n)_{\mathfrak{p}}}$ where $V_{(j_{1,\sigma},j_{2,\sigma})_\sigma}$ corresponds to the $(\mathbf{GL}_n \times \mathbf{GL}_n)_{\mathfrak{p}}$-representation $\sigma(\det(h_{\sigma,1})^{j_1}\det(h_{\sigma,2})^{j_2})$.

2. *Let* $\lambda_{\mathfrak{p}} = (\lambda_\sigma) \in X_0^*(\mathbf{T})_{\mathfrak{p}}$ *and* $j_{\mathfrak{p}} = (j_\sigma)$ *such that* $\forall \sigma \in \Sigma_{F_{\mathfrak{p}}^+}$ *we have* $-\lambda_{\sigma,n} \leq j_\sigma \leq -\lambda_{\sigma,n+1}$. *Then:*

$$v_{\lambda_{\mathfrak{p}},j_{\mathfrak{p}}} := r\left(\bigotimes_{\sigma \in \Sigma_{F_{\mathfrak{p}}}} v_{\lambda_\sigma,j_\sigma}\right) \in V_{\lambda_{\mathfrak{p}}}$$

such that $F_{\mathfrak{p}_\sigma}^+ v_{\lambda_{\mathfrak{p}},j_{\mathfrak{p}}} = V_{(-j_{\mathfrak{p}},w_{\mathfrak{p}}+j_{\mathfrak{p}})_{\mathfrak{p}}} \subseteq V_{\lambda_{\mathfrak{p}}}|_{(\mathbf{GL}_n \times \mathbf{GL}_n)_{\mathfrak{p}}}$ *where* $V_{(j_{1,\mathfrak{p}},j_{2,\mathfrak{p}})_{\mathfrak{p}}} := r(\otimes_{\sigma \in \Sigma_{F_{\mathfrak{p}}}} V_{(j_{1,\sigma},j_{2,\sigma})_\sigma})$.

3. *Let* $\underline{\lambda} = (\lambda_\sigma) \in X_0^*(\mathbf{T})$ *and* $\underline{j} = (j_\sigma)$ *such that* $\forall \sigma \in \Sigma_{F^+}$ *we have* $-\lambda_{\sigma,n} \leq j_\sigma \leq -\lambda_{\sigma,n+1}$. *Then:*

$$v_{\underline{\lambda},\underline{j}} := \bigotimes_{\sigma \in \Sigma_{F^+}} v_{\lambda_\sigma,j_\sigma} \in V_{\underline{\lambda}},$$

such that $(F^+ \otimes \mathbb{Q}_p) v_{\underline{\lambda},\underline{j}} = V_{(-\underline{j},\underline{w}+\underline{j})} \subseteq V_{\underline{\lambda}}|_{\mathrm{Res}_{F^+/\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_n)}$ *where* $V_{(\underline{j}_1,\underline{j}_2)} := \otimes_{\mathfrak{p}|p} V_{(j_{1,\mathfrak{p}},j_{2,\mathfrak{p}})_{\mathfrak{p}}}$.

Proof. Part 2 follows directly from 1, and part 3 similarly follows from 2. To prove part 1, we first observe:

$$\begin{aligned}
&v_{\sigma,0}^{\otimes \lambda_{\sigma,n+1}} \otimes \left(\bigotimes_{1 \leq i \leq n-1} v_{\sigma,i}^{\otimes(\lambda_{\sigma,i}-\lambda_{\sigma,i+1})}\right) \otimes v_{\sigma,n,1}^{\otimes(-\lambda_{\sigma,n+1}-j_\sigma)} \otimes v_{\sigma,n,2}^{\otimes(\lambda_{\sigma,n}+j_\sigma)} \\
&\in V_{\alpha_{\sigma,0}}^{\otimes \lambda_{\sigma,n+1}} \otimes \left(\bigotimes_{1 \leq i \leq n-1} V_{\alpha_{\sigma,i}}^{\otimes(\lambda_{\sigma,i}-\lambda_{\sigma,i+1})}\right) \otimes V_{\alpha_{\sigma,n}}^{\otimes(-j_\sigma-\lambda_{\sigma,n+1})} \otimes V_{\alpha_{\sigma,n}}^{\otimes(\lambda_{\sigma,n}+j_\sigma)} \\
&= V_{\alpha_{\sigma,0}}^{\otimes \lambda_{\sigma,n+1}} \otimes \left(\bigotimes_{1 \leq i \leq n-1} V_{\alpha_{\sigma,i}}^{\otimes(\lambda_{\sigma,i}-\lambda_{\sigma,i+1})}\right) \otimes V_{\alpha_{\sigma,n}}^{\otimes(\lambda_{\sigma,n}-\lambda_{\sigma,n+1})} \\
&= V_{\alpha_{\sigma,0}}^{\otimes \lambda_{\sigma,n+1}} \otimes \left(\bigotimes_{1 \leq i \leq n} V_{\alpha_{\sigma,i}}^{\otimes(\lambda_{\sigma,i}-\lambda_{\sigma,i+1})}\right),
\end{aligned}$$

and thus by definition of r combined with part 2 of proposition 3.1, we know $v_{\lambda_\sigma,j_\sigma} \in V_{\lambda_\sigma}$. It then remains to show that it is the claimed $(\mathbf{GL}_n \times \mathbf{GL}_n)_{\mathfrak{p}_\sigma}$-eigenvector which we can see from calculating the $(\mathbf{GL}_n \times \mathbf{GL}_n)_{\mathfrak{p}_\sigma}$-action. We have that $(h_{\mathfrak{p}_\sigma,1}, h_{\mathfrak{p}_\sigma,2}) \in (\mathbf{GL}_n \times \mathbf{GL}_n)_{\mathfrak{p}_\sigma}$ acts on $v_{\lambda_\sigma,j_\sigma}$ with:

$$\det(h_{\mathfrak{p}_\sigma,1})^{-\lambda_{\sigma,n+1}-j_\sigma} \cdot \det(h_{\mathfrak{p}_\sigma,2})^{\lambda_{\sigma,n}+j_\sigma} \det(h_{\mathfrak{p}_\sigma,1} \times h_{\mathfrak{p}_\sigma,2})^{\lambda_{\sigma,n+1}} = \det(h_{\mathfrak{p}_\sigma,1})^{-j_\sigma} \cdot \det(h_{\mathfrak{p}_\sigma,2})^{w_\sigma+j_\sigma}.$$

Thus $F_{\mathfrak{p}}^+ v_{\lambda_\sigma,j_\sigma} = V_{(-j_\sigma,w_\sigma+j_\sigma)_\sigma}$. □

3.2 p-adic interpolation of branching laws

We now turn to the task of enabling p-adic interpolation of the family of vectors from the last section. To do this, we have an essential change of language through the Borel–Weil–Bott theorem.

$$V_{\underline{\lambda}} \cong \mathrm{Ind}_{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{B}^-}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{GL}_{2n}}(\underline{\lambda}) := \{f \in \mathrm{Mor}(\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{GL}_{2n}, \mathbf{G}_a(\underline{\lambda})) : \forall b \in \mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{B}^- \ f(bg) = \underline{\lambda}(b)f(g)\}$$

And similarly in view of the decomposition $V_{\underline{\lambda}} = \otimes_{\mathfrak{p}|p} V_{\lambda_{\mathfrak{p}}} = r(\otimes_{\sigma \in \Sigma_{F^+}} V_{\lambda_\sigma})$.

$$V_{\lambda_\sigma} \cong \mathrm{Ind}_{\mathbf{B}_\sigma^-}^{\mathbf{GL}_{\sigma,2n}}(\lambda_\sigma) := \{f \in \mathrm{Mor}(\mathbf{GL}_{\sigma,2n}, \mathbf{G}_a(\lambda_\sigma)) : \forall b \in \overline{\mathbf{B}}_\sigma \ f(bg) = \lambda_\sigma(b)f(g)\}.$$

In light of this, the concrete correspondence between our earlier basis vectors and functions become

$$\begin{aligned}
v \in V_{\underline{\lambda}} &\leftrightarrow v : \mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{GL}_{2n}(\mathbb{Z}_p) \to F^+ \otimes \mathbb{Q}_p \text{ s.t. } v(bg) = \underline{\lambda}(b)v(g) \\
v_\sigma \in V_{\lambda_\sigma} &\leftrightarrow v_\sigma : \mathbf{GL}_{2n}(F_{\mathfrak{p}}^+) \to F_{\mathfrak{p}}^+ \text{ s.t. } v_\sigma(bg) = \lambda_\sigma(b)v_\sigma(g).
\end{aligned}$$

As such, the decomposition translates as a multiplicative product:

$$v = \bigotimes_{\mathfrak{p}|p} r\left(\prod_{\sigma\in\Sigma_{F_\mathfrak{p}^+}} v_\sigma\right) = r\left(\bigotimes_{\sigma\in\Sigma_{F^+}} v_\sigma\right) \leftrightarrow v(g) = \prod_{\mathfrak{p}|p}\left(\prod_{\sigma\in\Sigma_{F_\mathfrak{p}^+}} v_\sigma(g)\right) = \prod_{\sigma\in\Sigma_{F^+}} v_\sigma(g)$$

Using this correspondence, we can rewrite the vector from part 4 of proposition 1. to become

$$v_{\lambda_\sigma,j_\sigma}(g) := v_{\sigma,0}(g)^{\lambda_{\sigma,n+1}} \cdot \left(\prod_{1\le i\le n-1} v_{\sigma,i}(g)^{\lambda_{\sigma,i}-\lambda_{\sigma,i+1}}\right) \cdot \frac{v_{\sigma,n,2}(g)^{\lambda_{\sigma,n}}}{v_{\sigma,n,1}(g)^{\lambda_{\sigma,n+1}}} \cdot \left(\frac{v_{\sigma,n,2}(g)}{v_{\sigma,n,1}(g)}\right)^{j_\sigma}.$$

To p-adically interpolate the weights, we recall section 1.2.5, and we let $\Omega \subseteq \mathscr{W}_0^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{GL}_{2n}}(L)$ be an affinoid with the character $\chi_\Omega : \mathbf{T}(\mathbb{Z}_p) \to \mathcal{O}_\Omega^\times$. Similarly to the weights, it decomposes into $\chi_\Omega = (\chi_{\Omega,\mathfrak{p}})_{\mathfrak{p}|p}$ where $\chi_{\Omega,\mathfrak{p}} : \mathbf{T}_{2n}(\mathcal{O}_{F_\mathfrak{p}^+}) \to \mathcal{O}_\Omega^\times$ and this further decomposes $\chi_{\Omega,\mathfrak{p}} = (\chi_{\Omega,\sigma,i})_{\sigma\in\Sigma_{F_\mathfrak{p}^+},1\le i\le 2n}$. Following the naive approach, we will seek to replace the weights $\lambda_{\sigma,i} : z \mapsto z^{\lambda_{\sigma,i}}$ with $z \mapsto \chi_{\Omega,\sigma,i}(z)$. For consistency, we write $z^{\chi_{\Omega,\sigma,i}} := \chi_{\Omega,\sigma,i}(z)$. However, it is not a priori clear that this replacement is well defined because $\mathrm{Im}(v_{\lambda_\sigma,j_\sigma}) \subseteq F_\mathfrak{p}^+$ and we need $\mathrm{Im}(v_{\lambda_\sigma,j_\sigma}) \subseteq \mathcal{O}_{F_{\mathfrak{p}\sigma}^+}^\times$ to ensure it embeds into the domain of $\chi_{\Omega,\sigma,i}$. To do this, we will see that the sub-monoid

$$\mathbf{N}_\mathbf{B}^{\beta_\mathfrak{p}}(\mathcal{O}_{F_\mathfrak{p}^+}) := \left\{g \in \mathbf{N}_\mathbf{B}(\mathcal{O}_{F_\mathfrak{p}^+}) : g \equiv u_\mathfrak{p} \mod \mathfrak{p}^{\beta_\mathfrak{p}}\right\} \subseteq \mathbf{N}_\mathbf{B}(\mathcal{O}_{F_\mathfrak{p}^+}) \tag{3.5}$$

and its extension $\mathrm{Iw}_{\mathbf{B},\mathfrak{p}}^\beta := (\mathrm{Iw}_{\mathbf{B},\mathfrak{p}} \cap \mathbf{N}_\mathbf{B}^-(\mathcal{O}_{F_\mathfrak{p}^+})) \cdot \mathbf{L}_\mathbf{P}(\mathcal{O}_{F_\mathfrak{p}^+}) \cdot \mathbf{N}_\mathbf{B}^{\beta_\mathfrak{p}}(\mathcal{O}_{F_\mathfrak{p}^+})$ allow us to control the image.

Lemma 3.3. *Let $\mathfrak{p} \mid p$ be a prime ideal, $\lambda_\mathfrak{p} \in X_0^*(\mathbf{T})_\mathfrak{p}$ and $j_\mathfrak{p} = (j_\sigma) \in \mathrm{Crit}(\lambda_\mathfrak{p})$. Then for $\beta_\mathfrak{p} \in \mathbb{Z}_{\ge 0}$ there exists $c \in F_\mathfrak{p}^+$ such that for $v' := c \cdot v_{\lambda_\mathfrak{p},j_\mathfrak{p}}$ we have*

$$v'(\mathbf{N}_\mathbf{B}^{\beta_\mathfrak{p}}(\mathcal{O}_{F_\mathfrak{p}^+})) \subseteq 1 + \mathfrak{p}^{\beta_\mathfrak{p}}\mathcal{O}_{F_\mathfrak{p}^+}.$$

Proof. We see that $\lambda_\mathfrak{p}(\mathbf{B}_{\mathbf{GL}_{2n}}^-(\mathcal{O}_{F_\mathfrak{p}^+})) \in \mathcal{O}_{F_\mathfrak{p}^+}^\times$ and $\prod_{\sigma\in\Sigma_{F_\mathfrak{p}^+}} \sigma(\det(\mathbf{GL}_n(\mathcal{O}_{F_\mathfrak{p}^+}))^{j_{1,\sigma}} \cdot \det(\mathbf{GL}_n(\mathcal{O}_{F_\mathfrak{p}^+}))^{j_{2,\sigma}}) \in \mathcal{O}_{F_\mathfrak{p}^+}$, so we have

$$v(\mathbf{B}_{\mathbf{GL}_{2n}}^-(\mathcal{O}_{F_\mathfrak{p}^+}) \cdot u \cdot (\mathbf{GL}_n \times \mathbf{GL}_n)(\mathcal{O}_{F_\mathfrak{p}^+})) \in \mathcal{O}_{F_\mathfrak{p}^+}^\times \cdot v(u).$$

Assume for contradiction that $v(u) = 0$. Then $v(\mathbf{B}_{\mathbf{GL}_{2n}}^-(\mathcal{O}_{F_\mathfrak{p}^+}) \cdot u \cdot (\mathbf{GL}_n \times \mathbf{GL}_n)(\mathcal{O}_{F_\mathfrak{p}^+})) = 0$. We may recall $\mathbf{GL}_{2n}/(\mathbf{GL}_n \times \mathbf{GL}_n)$ is a spherical variety as $\mathbf{B}_{\mathbf{GL}_{2n}}^+ \cdot u \cdot (\mathbf{GL}_n \times \mathbf{GL}_n)$ is dense and open in $\mathbf{GL}_{2n}$. Since v is continuos, this implies that $v(\mathbf{GL}_{2n}(\mathcal{O}_{F_\mathfrak{p}^+})) = 0$ and thus trivial which contradicts our assumption. Consequently, there exists $c \in F_\mathfrak{p}^+$ such that $v'(u) = 1$. By lemma 11.5 of [BDGJW25] where we replace the p-adic integers $\mathbf{Z}_p$ with $\mathcal{O}_{F_\mathfrak{p}^+}$ and p with the uniformiser ϖ of $F_\mathfrak{p}^+$, we have for all $X \in M_{2n}(\mathcal{O}_{F_\mathfrak{p}^+})$:

$$v'\begin{pmatrix} I_n & w_n + \varpi^{\beta_\mathfrak{p}} X \\ 0 & I_n \end{pmatrix} \in 1 + \pi^{\beta_\mathfrak{p}}\mathcal{O}_{F_\mathfrak{p}^+}.$$

Let $n \in \mathbf{N}_\mathbf{B}^{\beta_\mathfrak{p}}(\mathcal{O}_{F_\mathfrak{p}^+})$, then there are $A, B \in \mathbf{N}_{\mathbf{B}_{\mathbf{GL}_n}}(\mathcal{O}_{F_\mathfrak{p}^+})$ such that $A \equiv I_n \equiv B \mod \mathfrak{p}^{\beta_\mathfrak{p}}$ and $Y \in M_n(\mathcal{O}_{F_\mathfrak{p}^+})$ such that:

$$n = \begin{pmatrix} A & w_n + \varpi^{\beta_\mathfrak{p}} Y \\ 0 & B \end{pmatrix}$$

If we set $X = Y - \varpi^{-\beta_\mathfrak{p}} \cdot w_n(B - I_n)B^{-1} \in M_n(\mathcal{O}_{F_\mathfrak{p}^+})$, then we have:

$$v'(n) = \prod_{\sigma\in\Sigma_{F_\mathfrak{p}}} (\det(A)^{j_{1,\sigma}} \cdot \det(B)^{j_{2,\sigma}}) \cdot v'\left(\begin{pmatrix} I_n & w_n + \varpi^{\beta_\mathfrak{p}} X \\ 0 & I_n \end{pmatrix}\right) \in 1 + \mathfrak{p}^{\beta_\mathfrak{p}}\mathcal{O}_{F_\mathfrak{p}^+}.$$

□

Remark 3.4. *The matrix u has been chosen to make the aforementioned lemma work nicely with $\mathbf{H}$. Other choices of u is possible up to proving a corresponding result.*

We then take note of the immediate consequence of this.

Lemma 3.5. *With appropriate rescaling of the $(\mathbf{GL}_n \times \mathbf{GL}_n)_\sigma$-eigenvectors from part 2 of proposition 3.1, we have a well defined function the function $w_{\chi_\sigma} : \mathbf{N}_{\mathbf{B}}^{\beta_\mathfrak{p}}(\mathcal{O}_{F_\mathfrak{p}^+}) \to \mathcal{O}_\Omega$ defined as*

$$w_{\chi_\sigma}(g) := v_{\sigma,0}(g)^{\chi_{\sigma,n+1}} \cdot \left(\prod_{1 \leq i \leq n-1} v_{\sigma,i}(g)^{\chi_{\sigma,i} - \chi_{\sigma,i+1}} \right) \cdot \frac{v_{\sigma,n,2}(g)^{\chi_{\sigma,n}}}{v_{\sigma,n,1}(g)^{\chi_{\sigma,n+1}}}.$$

Proof. This follows as an immediate application of lemma 3.3. □

Thus for the cases $\chi_{\sigma,i}(z) = (z \mapsto z^{\lambda_{\sigma,i}})$, we have $w_{\chi_\sigma}(g) \cdot \left(\frac{v_{\sigma,n,2}(g)}{v_{\sigma,n,1}(g)}\right)^{j_\sigma} = v_{\lambda_\sigma, j_\sigma}(g)$. It remains to incorporate the infinity type. To do this, we have to handle the support conditions delicately. We recall

$$x \in \mathscr{U}_\beta := j(1 + p^\beta \mathcal{O}_F \otimes \mathbb{Z}_p) \subset \Gamma^{\mathrm{ac}}_{p^\infty \mathfrak{R}}.$$

from subsection 1.2.15, and we define $\mathcal{A}(\mathscr{U}_\beta, L) \subseteq \mathcal{A}(\Gamma^{\mathrm{ac}}_{p^\infty \mathfrak{R}}, L)$ as the subspace of functions supported on $\mathscr{U}_\beta$. We then observe that the Iwahoric decomposition of $\mathrm{Iw}_{\mathbf{B}}^\beta$ yields

$$\mathcal{A}_\Omega^{\mathbf{B},\beta} \cong \mathcal{A}(\mathbf{N}^\beta_{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{B}}(\mathbb{Z}_p), \mathcal{O}_\Omega). \tag{3.6}$$

where $\mathcal{A}_\Omega^{\mathbf{B},\beta}$ is defined as the subset of $\mathcal{A}_\Omega^{\mathbf{B}}$ with support on $\mathrm{Iw}_{\mathbf{P}}^\beta$. Lastly, we define the $\mathbf{H}(\mathbb{A}_\mathbb{Q})$ action on $f \in \mathcal{A}(\Gamma^{\mathrm{ac}}_{p^\infty \mathfrak{R}}, \mathcal{O})$ as

$$(h * f)(x) := \eta_{[p]}(\det h_2) f\left(\frac{\det h_1}{\det h_2} x\right) \tag{3.7}$$

and recall the linear map $\iota = \prod_\mathfrak{p} \sigma_\mathfrak{p} : F^+ \otimes \mathbb{Q}_p \to L$ which we will implicitly use.

Proposition 3.6. *Let $f \in \mathcal{A}(\mathscr{U}_\beta, \mathcal{O}_\Omega)$ be a locally analytic function. Then:*

1. *With appropriate scaling of the $\mathrm{Res}_{F^+/\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_n)$-eigenvectors from part 2 of proposition 3.1, the following is a well-defined function $v_\Omega : \mathbf{N}^\beta_{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{B}}(\mathbb{Z}_p) \to \mathcal{O}_\Omega$*

$$v_\Omega^\beta(g) := \left(\prod_{\sigma \in \Sigma_{F^+}} w_{\chi_\sigma}(g) \right) \cdot f\left(\oplus_{\mathfrak{p}|p} \frac{v_{\sigma_\mathfrak{p},n,1}(g)}{v_{\sigma_\mathfrak{p},n,2}(g)} \right)$$

   which through equation 3.6 can be viewed as a function

$$v_\Omega^\beta : \mathcal{A}(\mathscr{U}_\beta, \mathcal{O}_\Omega) \to \mathcal{A}_\Omega^{\mathbf{B},\beta}.$$

2. *Let $\chi \in \mathrm{Crit}(\underline{\lambda})[p^\beta \mathfrak{R}]$. Then we have*

$$sp_{\underline{\lambda}}(v_\Omega^\beta(\chi_{[p]})) = v_{\underline{\lambda},\underline{j}},$$

3. *The function v_Ω^β is $\mathrm{Iw}_{\mathbf{B}}^\beta$-equivariant.*

Proof. We now prove the statements sequentially.

1. This is direct consequence of lemma 3.5 with further scaling the remaining $\mathrm{Res}_{F^+/\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_n)$-eigenvectors as in lemma 3.3.

2. The second part of the proposition follows from a direct calculation. We first note that via equation 3.6, it suffices to only show this for $g \in \mathbf{N}^{\beta}_{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{B}}(\mathbb{Z}_p)$. Then for each prime $\mathfrak{p} \mid p$, we consider the embeddings $\sigma : F^+ \hookrightarrow F^+_{\mathfrak{p}}$ which conjugate under the action of $\mathrm{Gal}(F^+_{\mathfrak{p}}/\mathbb{Q}_p) \cong G_{\mathfrak{p}}$. We then calculate:

$$\begin{aligned}
\mathrm{sp}_{\underline{\lambda}}(v^{\beta}_{\Omega}(\chi_{[p]}))(g) &= \mathrm{sp}_{\underline{\lambda}}\left(\left(\prod_{\sigma\in\Sigma_{F^+}} w_{\chi_\sigma}(g)\right)\cdot\chi_{[p]}\left(\bigoplus_{\mathfrak{p}|p}\frac{v_{\sigma_{\mathfrak{p}},n,1}(g)}{v_{\sigma_{\mathfrak{p}},n,2}(g)}\right)\right)\\
&= \mathrm{sp}_{\underline{\lambda}}\left(\prod_{\sigma\in\Sigma_{F^+}} w_{\chi_\sigma}(g)\right)\cdot\omega_p^{-\underline{j}}\left(\bigoplus_{\mathfrak{p}|p}\frac{v_{\sigma_{\mathfrak{p}},n,1}(g)}{v_{\sigma_{\mathfrak{p}},n,2}(g)}\right)\\
&= \mathrm{sp}_{\underline{\lambda}}\left(\prod_{\sigma\in\Sigma_{F^+}} w_{\chi_\sigma}(g)\right)\cdot\prod_{\mathfrak{p}|p}\sigma_{\mathfrak{p}}\left(\prod_{\tau\in G_{\mathfrak{p}}}\tau\left(\frac{v_{\sigma_{\mathfrak{p}},n,2}(g)}{v_{\sigma_{\mathfrak{p}},n,1}(g)}\right)^{j_{\sigma_{\mathfrak{p}}\tau}}\right)\\
&= \prod_{\sigma\in\Sigma_{F^+}}\mathrm{sp}_{\underline{\lambda}}(w_{\chi_\sigma})(g)\cdot\left(\frac{v_{\sigma,n,2}(g)}{v_{\sigma,n,1}(g)}\right)^{j_\sigma} = \prod_{\sigma\in\Sigma_{F^+}} v_{\lambda_\sigma,j_\sigma}(g) = v_{\underline{\lambda},\underline{j}}(g).
\end{aligned}$$

In the above, equality 1. follows from definition of v_Ω, equality 2. follows from the discussion of section 1.2.5 of $\chi_{[p]}$ combined with noting that $\chi_{[p]}(x) = w_p^{\underline{j}}(x)$ for $x \in \mathscr{U}_\beta$, equality 3. follows from expanding the definition of $w_p^{-\underline{j}}$, equality 4. follows from part 3 of proposition 3.1 and reordering of factors, and the remaining equalities follow from the definition of $v_{\underline{\lambda},\underline{j}}$.

3. We only need to consider $g \in N^{\beta}_{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{B}}(\mathbb{Z}_p)$ and $h \in \mathrm{Iw}^{\beta}_{\mathbf{B}_{\mathrm{Res}_{F^+/\mathbb{Q}}(\mathbf{GL}_n\times\mathbf{GL}_n)}}$ and observe

$$\begin{aligned}
v^{\beta}_{\Omega}(h*f)(g) &= \prod_{\sigma\in\Sigma_{F^+}} w_{\chi_\sigma}(g)\cdot(h*f)\left(\bigoplus_{\mathfrak{p}|p}\frac{v_{\sigma_{\mathfrak{p}},n,1}(g)}{v_{\sigma_{\mathfrak{p}},n,2}(g)}\right)\\
&= \prod_{\sigma\in\Sigma_{F^+}} w_{\chi_\sigma}(g)\cdot\eta_{[p]}(\det h_2^{-1})\cdot f\left(\bigoplus_{\mathfrak{p}|p}\frac{\det(h_1)\cdot v_{\sigma_{\mathfrak{p}},n,1}(g)}{\det(h_2)\cdot v_{\sigma_{\mathfrak{p}},n,2}(g)}\right)\\
&= \prod_{\sigma\in\Sigma_{F^+}} w_{\chi_\sigma}(g)\cdot\omega_p^{\underline{w}}(\det h_2)\cdot f\left(\bigoplus_{\mathfrak{p}|p}\frac{\det(h_1)\cdot v_{\sigma_{\mathfrak{p}},n,1}(g)}{\det(h_2)\cdot v_{\sigma_{\mathfrak{p}},n,2}(g)}\right)\\
&= \prod_{\sigma\in\Sigma_{F^+}} w_{\chi_\sigma}(gh)\cdot f\left(\bigoplus_{\mathfrak{p}|p}\frac{v_{\sigma_{\mathfrak{p}},n,1}(gh)}{v_{\sigma_{\mathfrak{p}},n,2}(gh)}\right)
\end{aligned}$$

where the last equality uses that $\chi_{\sigma,i}\cdot\chi_{\sigma,2n+1-i} = w_\sigma$. We may observe that $g \in \mathrm{Iw}^{\beta}_{\mathbf{B}}$ and so $gh \in \mathrm{Iw}^{\beta}_{\mathbf{B}}$. By the parahoric decomposition from equation 3.6, it follows that there exists

$b' \in \mathbf{N}^-_{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{B}}(p\mathbb{Z}_p)\cdot\mathbf{T}(\mathbb{Z}_p)$ and $g' \in \mathbf{N}^\beta_{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{B}}(\mathbb{Z}_p)$ such that $gh = b'g'$. Thus:

$$\begin{aligned}(h * v^\beta_\Omega(f))(g) = v_\Omega(f)(gh) = v^\beta_\Omega(f)(b'g') &= \chi_\Omega(b')\cdot v^\beta_\Omega(f)(g') \\ &= \prod_{\sigma\in\Sigma_{F^+}} \chi_\sigma(b')\cdot w_{\chi_\sigma}(g')\cdot f\left(\bigoplus_{\mathfrak{p}|p}\frac{v_{\sigma_\mathfrak{p},n,1}(g')}{v_{\sigma_\mathfrak{p},n,2}(g')}\right) \\ &= \prod_{\sigma\in\Sigma_{F^+}} w_{\chi_\sigma}(b'g')\cdot f\left(\bigoplus_{\mathfrak{p}|p}\frac{v_{\sigma_\mathfrak{p},n,1}((b')^{-1}gh)}{v_{\sigma_\mathfrak{p},n,2}((b')^{-1}gh)}\right) \\ &= \prod_{\sigma\in\Sigma_{F^+}} w_{\chi_\sigma}(gh)\cdot f\left(\bigoplus_{\mathfrak{p}|p}\frac{v_{\sigma_\mathfrak{p},n,1}(gh)}{v_{\sigma_\mathfrak{p},n,2}(gh)}\right)\end{aligned}$$

The result follows from observing that the last step of the two proceeding calculations agree.

□

3.3 Branching laws for distributions

It now remains to dualise the above and show that it assembles into the desired result. For parabolic subgroups $\mathbf{P}$, we define $\mathcal{D}^{\mathbf{P},\beta}_\Omega$ as the subset of $\mathcal{D}^{\mathbf{P}}_\Omega$ with support on $\mathrm{Iw}^\beta_{\mathbf{P}}$, that is $\mu(f) = \mu(f|_{\mathrm{Iw}^\beta_{\mathbf{P}}})$ where $\mathcal{A}^{\mathbf{P},\beta}_\Omega$ is defined like $\mathcal{A}^{\mathbf{B},\beta}_\Omega$ as in equation 3.6, and we define an auxiliary monoid

$$J^\beta_{\mathbf{P}} := \mathbf{H}(\mathbb{Z}_p)\cap u^{-1}\,\mathrm{Iw}_{\mathbf{P}}\,. \tag{3.8}$$

We further recall from equation 1.24 that we have projective $p_{\mathbf{P}'/\mathbf{P}}$ and injective $\iota_{\mathbf{P}'/\mathbf{P}}$ maps on the coefficient modules for parabolics $\mathbf{P}\subseteq\mathbf{P}'$ with dual maps $p^\vee_{\mathbf{P}'/\mathbf{P}}, \iota^\vee_{\mathbf{P}'/\mathbf{P}}$ for the distributions, and these descend to $\mathcal{D}^{\mathbf{P},\beta}_\Omega$ and $\mathcal{A}^{\mathbf{P},\beta}_\Omega$ for relevant parabolics.

Theorem 3.7. *Let $\mathbf{P}$ be a relevant parabolic, $K\subseteq\mathbf{G}(\mathbb{A}_\mathbb{Q})$ an open compact subgroup, $\Omega\subseteq\mathscr{W}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}_{K,0}(L)$ an affinoid and $\beta=(\beta_\mathfrak{p})_{\mathfrak{p}|p}\in\mathbb{Z}^{\mathfrak{p}|p}_{\geq 0}$. Then the map*

$$\kappa^\beta_\Omega := (v^\beta_\Omega)^\vee\circ p^\vee_{\mathbf{P}/\mathbf{B}} : \mathcal{D}^{\mathbf{P},\beta}_\Omega \to \mathcal{D}(\Gamma^{ac}_{p^\infty\mathfrak{R}},\mathcal{O}_\Omega)$$

yields the following commutative diagram for all classical $\underline{\lambda}\in\Omega^{cl}$ and anti-cyclotomic Hecke characters $\chi\in\mathrm{Crit}(\underline{\lambda})[p^\beta\mathfrak{R}]$

$$\begin{array}{ccc}
\mathcal{D}^{\mathbf{P},\beta}_\Omega & \xrightarrow{\kappa^\beta_\Omega} & \mathcal{D}(\Gamma^{ac}_{p^\infty\mathfrak{R}},\mathcal{O}_\Omega) \\
\downarrow{\scriptstyle sp_{\underline{\lambda}}} & & \downarrow{\scriptstyle sp_{\underline{\lambda}}} \\
\mathcal{D}^{\mathbf{P},\beta}_{\underline{\lambda}} & \xrightarrow{\kappa^\beta_{\underline{\lambda}}} & \mathcal{D}(\Gamma^{ac}_{p^\infty\mathfrak{R}},L) \\
\downarrow{\scriptstyle \iota^\vee_{\mathbf{G}/\mathbf{P}}} & & \downarrow{\scriptstyle \mu\mapsto\mu(\chi_{[p]})} \\
V^\vee_{\underline{\lambda}}(L) & \xrightarrow{\kappa_{\underline{\lambda},j_\chi}} & L
\end{array}$$

where we denote κ^β_Ω as $\kappa^\beta_{\underline{\lambda}}$ when $\Omega=\{\underline{\lambda}\}$ and all the maps are $J^\beta_{\mathbf{P}}$-equivariant.

Proof. We will prove this result by considering the following diagram.

$$\begin{array}{ccccc}
\mathcal{D}_{\Omega}^{\mathbf{P},\beta} & \xrightarrow{p^{\vee}_{\mathbf{P}/\mathbf{B}}} & \mathcal{D}_{\underline{\lambda}}^{\mathbf{B},\beta} & \xrightarrow{\kappa_{\Omega}} & \mathcal{D}(\Gamma^{\mathrm{ac}}_{p^{\infty}\mathfrak{R}}, \mathcal{O}_{\Omega}) \\
\downarrow{\scriptstyle \mathrm{sp}_{\underline{\lambda}}} & & \downarrow{\scriptstyle \mathrm{sp}_{\underline{\lambda}}} & & \downarrow{\scriptstyle \mathrm{sp}_{\underline{\lambda}}} \\
\mathcal{D}_{\underline{\lambda}}^{\mathbf{P},\beta} & \xrightarrow{p^{\vee}_{\mathbf{P}/\mathbf{B}}} & \mathcal{D}_{\underline{\lambda}}^{\mathbf{B},\beta} & \xrightarrow{\kappa_{\underline{\lambda}}} & \mathcal{D}(\Gamma^{\mathrm{ac}}_{p^{\infty}\mathfrak{R}}, L) \\
\downarrow{\scriptstyle \iota^{\vee}_{\mathbf{G}/\mathbf{P}}} & & \downarrow{\scriptstyle \iota^{\vee}_{\mathbf{G}/\mathbf{B}}} & & \downarrow{\scriptstyle \mu \mapsto \mu(\chi_{[p]})} \\
V^{\vee}_{\underline{\lambda}}(L) & \xrightarrow{\mathrm{Id}} & V^{\vee}_{\underline{\lambda}}(L) & \xrightarrow{\kappa_{\underline{\lambda},\underline{j}_{\chi}}} & L
\end{array}$$

For now, we claim the left side is commutative and $J_{\mathbf{P}}^{\beta}$-equivariant leaving us with the right hand side.

1. We first observe that the upper right square of the diagram commutes by the definition of the $sp_{\underline{\lambda}}$ map. Consequently, it suffices to show that the lower middle square commutes. Let $\mu \in \mathcal{D}_{\lambda}^{\mathbf{B},\beta}$ and observe:

$$\begin{aligned}
\kappa_{\underline{\lambda}}^{\beta}(\mu)(\chi_{[p]}) &= \int_{\Gamma^{\mathrm{ac}}_{p^{\infty}\mathfrak{R}}} \chi_{[p]} \cdot d\kappa_{\lambda}^{\beta}(\mu) \\
&= \int_{\mathrm{Iw}_{\mathbf{B}}^{\beta}} v_{\lambda}^{\beta}(\chi_{[p]}) \cdot d\mu \\
&= \int_{\mathrm{Iw}_{\mathbf{B}}^{\beta}} v_{\underline{\lambda},\underline{j}} \cdot d\mu \\
&= \int_{\mathrm{Iw}_{\mathbf{B}}^{\beta}} v_{\underline{\lambda},\underline{j}} \cdot dr_{\lambda}(\mu) = \kappa_{\underline{\lambda},\underline{j}} \circ \iota^{\vee}_{\mathbf{G}/\mathbf{B}}(\mu).
\end{aligned}$$

   where we change notation in equality 1, we apply the definition of $\kappa_{\underline{\lambda}}^{\beta}$ in equality 2, use part 2 of proposition 3.6 in equality 3 and the definition of $\iota^{\vee}_{\mathbf{G}/\mathbf{B}}$ in equality 4.

2. We then check that the right hand side is $J_{\mathbf{P}}^{\beta}$-equivariant. We consider the vertical maps first, and see that the statement holds for $sp_{\underline{\lambda}}$ and $\iota^{\vee}_{\mathbf{G}/\mathbf{B}}$ by definition. To consider the evaluation map $\mu \mapsto \mu(\chi_{[p]})$, we let $\mu \in \mathcal{D}(\Gamma^{\mathrm{ac}}_{p^{\infty}\mathfrak{R}}, L)$, $h = (h_1, h_2) \in J_{\mathbf{P}}^{\beta}$ and claim that:

$$h * \chi_{[p]} = \omega_p^{-\underline{j}}(\det(h_1)) \cdot \omega_p^{\underline{w}+\underline{j}}(\det(h_2)) \cdot \chi_{[p]} \tag{3.9}$$

   With this claim in mind, we observe:

$$\begin{aligned}
(h * \mu)(\chi_{[p]}) &= \mu(h^{-1} * \chi_{[p]}) \\
&= \mu(\omega_p^{\underline{j}}(\det(h_1)) \cdot \omega_p^{-\underline{j}-\underline{w}}(\det(h_2)) \cdot \chi_{[p]}) \\
&= \omega_p^{\underline{j}}(\det(h_1)) \cdot \omega_p^{-\underline{j}-\underline{w}}(\det(h_2)) \cdot \mu(\chi_{[p]}) \\
&= h * \mu(\chi_{[p]})
\end{aligned}$$

   where the first equality arises from the dualisation of the action, the second equality comes from the claim of equation 3.9, the third equality is linearity and the last equality is the definition of the $\mathrm{Res}_{F^+/\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_n)$-action on $\mu(\chi_{[p]}) \in V^{\vee}_{(-\underline{j},\underline{j}+\underline{w})}$. Regarding the claim of equation 3.9, we observe:

$$\begin{aligned}
h * \chi_{[p]}(x) &= \eta_{[p]}(\det h_2)\chi_{[p]}(\det(h_1 h_2^{-1}) \cdot x) \\
&= \omega_p^{\underline{w}}(\det(h_2))\omega_p^{\underline{j}}(\det(h_1 h_2^{-1})) \cdot \chi_{[p]}(x) \\
&= \omega_p^{-\underline{j}}(\det(h_1)) \cdot \omega_p^{\underline{w}+\underline{j}}(\det(h_2)) \cdot \chi_{[p]}(x)
\end{aligned}$$

where the first equality is the defined action from 3.7 and the rest follows from applying the observation $\chi_{[p]}(x) = \omega_p^{\underline{j}}(x)$ for $x \equiv 1 \mod p^\beta$ where $x \in \mathbb{A}_{F^+}$. For κ_λ and κ_Ω, this follows from part 3 of proposition 3.6, and for $\kappa_{\underline{\lambda},\underline{j}}$ we observe for $\mu \in V_\lambda^\vee(L)$ similarly to above:

$$\begin{aligned}(h*\mu)(v_{\underline{\lambda},\underline{j}}) &= \mu(h^{-1} * v_{\underline{\lambda},\underline{j}}) \\ &= \mu(\omega_p^{\underline{j}}(\det(h_1)) \cdot \omega_p^{-\underline{j}-\underline{w}}(\det(h_2)) \cdot v_{\underline{\lambda},\underline{j}}) \\ &= \omega_p^{\underline{j}}(\det(h_1)) \cdot \omega_p^{-\underline{j}-\underline{w}}(\det(h_2)) \cdot \mu(v_{\underline{\lambda},\underline{j}}) \\ &= h * \mu(v_{\underline{\lambda},\underline{j}})\end{aligned}$$

where the first equality comes from the dualisation of the action, the second equality comes from the $\mathrm{Res}_{F^+/\mathbb{Q}}(\mathbf{GL}_n \times \mathbf{GL}_n)$-action on $v_{\underline{\lambda},\underline{j}} \in V_{(-\underline{j},\underline{w}+\underline{j})}$ which we know from lemma 3.2 and the rest is as above.

It then remains to prove the first claims for the left side of the commutative diagram. The top left square follows directly from the definition of $\mathrm{sp}_{\underline{\lambda}}$. For the bottom left square, we may observe that

$$\iota^\vee_{\mathbf{G}/\mathbf{B}} \circ p^\vee_{\mathbf{P}/\mathbf{B}} = \iota^\vee_{\mathbf{G}/\mathbf{P}} \circ \iota^\vee_{\mathbf{P}/\mathbf{B}} \circ p^\vee_{\mathbf{P}/\mathbf{B}} = \iota^\vee_{\mathbf{G}/\mathbf{P}}. \tag{3.10}$$

□

4 Interpolation of the p-adic periods

Equipped with a branching law, we now turn to lifting our previous evaluation map and building the desired locally analytic distribution.

4.1 Overconvergent evaluation maps

Before we continue, we first narrow down the open compact subgroup. Let $K^p \subseteq \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^p)$ be an open compact subgroup, $\mathbf{P}$ be a relevant parabolic and define $K_f^{\mathbf{P}} := K^p \operatorname{Iw}_{\mathbf{P}}$. For $\beta \in \mathbb{Z}_{\geq 1}^{\mathfrak{p}|p}$, we then define $K_{\mathfrak{p}}^{\mathbf{P},\beta_{\mathfrak{p}}} := u_{\mathfrak{p}}^{-1} t_{\mathbf{P},\mathfrak{p}}^{\beta_{\mathfrak{p}}} \operatorname{Iw}_{\mathfrak{P},\mathfrak{p}} t_{\mathbf{P},\mathfrak{p}}^{-\beta_{\mathfrak{p}}} u_{\mathfrak{p}}$, $K_p^{\mathbf{P},\beta} = \prod_{\mathfrak{p}|p} K_{\mathfrak{p}}^{\mathbf{P},\beta_{\mathfrak{p}}}$ and $K_f^{\mathbf{P},\beta} = K^p K_p^{\mathbf{P},\beta}$. With these open compacts, we define the following auxiliary open compact subgroups for our computations

1. $L^p := \{h \in \mathbf{H}(\mathbb{A}_{\mathbb{Q},f}^p) : h \equiv 1 \mod \mathfrak{R}\}$,
2. $L_{\mathbf{P},\mathfrak{p}}^{\beta_{\mathfrak{p}}} := \mathbf{H}(\mathbb{Z}_p) \cap K_{\mathfrak{p}} \cap u_{\mathfrak{p}} t_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} \operatorname{Iw}_{\mathbf{P}} t_{\mathfrak{p}}^{-\beta_{\mathfrak{p}}} u_{\mathfrak{p}}^{-1}$,
3. $L_{\mathbf{P},p}^{\beta} := \prod_{\mathfrak{p}|p} L_{\mathfrak{p}}^{\beta_{\mathfrak{p}}}$ and $L_{\mathbf{P}}^{\beta} := L_{\mathbf{P}}^p L_{\mathbf{P},p}^{\beta} \mathbf{G}(\mathbb{R})$,

for $\mathfrak{R}$ containing the kernel large enough such that $L^p \subseteq K^p \cap \mathbf{H}(\mathbb{A}_{\mathbb{Q}})$. Then $\prod_{\mathfrak{p}|p} L_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} \subseteq J_{\mathbf{P}}^{\beta}$.

Lemma 4.1. *Let $\mathbf{P}$ be a relevant parabolic, $K_f^{\mathbf{P},\beta}$ and $L_{\mathbf{P}}^{\beta}$ as above and $\Omega \subseteq \mathscr{W}_{K,0}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}(L)$ an affinoid. Then the following map is well defined*

$$Ev_\Omega^\beta(\mathcal{F}) : M_p(\mathcal{D}_\Omega^{\mathbf{P},\beta}, L_{\mathbf{P}}^\beta) \to \mathcal{D}(\Gamma_{p^\infty\mathfrak{R}}^{ac}, \mathcal{O}_\Omega); \mathcal{F} \mapsto \mathrm{Vol}(L_{\mathbf{P}}^\beta) \sum_{i=1}^{r} \frac{h_i * (\kappa_\Omega^\beta \circ \mathcal{F}(h_i))}{|\mathbf{H}(\mathbb{Q}) \cap h_i L_{\mathbf{P}}^\beta h_i^{-1}|} \tag{4.1}$$

where h_i are representatives of $[\mathbf{H}]/L_{\mathbf{P}}^\beta$, and we have for all classical weights $\underline{\lambda} \in \Omega$ and $\chi \in \mathrm{Crit}(\underline{\lambda})[p^\beta\mathfrak{R}]$ that

$$sp_{\underline{\lambda}}(Ev_\Omega^\beta(\mathcal{F})(\chi_{[p]})) = Ev_{\underline{\lambda},\chi}^p(sp_{\underline{\lambda}}(\mathcal{F})). \tag{4.2}$$

Proof. This proof has two parts: that the function is well-defined and that it yields the specialisation. To check that it is well-defined, we need to show that the definition is independent of the representatives. Given the representatives h_i, we will consider $h_i k_i$ for some $k_i \in L^\beta_{\mathbf{P}}$, and we will let $k_i = k_{i,p}k_i^p$ with $k_{i,p} \in L^\beta_{\mathbf{P},p}$ and $k_i^p \in L^p_{\mathbf{P}}$. Then we have:

$$\begin{aligned}\sum_{i=1}^{r} \frac{(h_i k_i) * (\kappa^\beta_\Omega \circ \mathcal{F}(h_i k_i))}{|\mathbf{H}(\mathbb{Q}) \cap h_i k_i L^\beta_{\mathbf{P}} k_i^{-1} h_i^{-1}|} &= \sum_{i=1}^{r} \frac{(h_i k_{i,p}) * (\kappa^\beta_\Omega \circ \mathcal{F}(h_i k_{i,p}))}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^\beta_{\mathbf{P}} h_i^{-1}|} \\ &= \sum_{i=1}^{r} \frac{h_i * (\kappa^\beta_\Omega (k_{i,p} k_{i,p}^{-1} \mathcal{F}(h_i)))}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^\beta_{\mathbf{P}} h_i^{-1}|} \\ &= \sum_{i=1}^{r} \frac{h_i * (\kappa^\beta_\Omega \circ \mathcal{F}(h_i))}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^\beta_{\mathbf{P}} h_i^{-1}|}.\end{aligned}$$

where we use in the first equality that $k_i^p * \mu = \mu$ and that κ^β_Ω is $\mathrm{Iw}^\beta_{\mathbf{P}}$-equivariant from theorem 3.7 in the second. The last is a subtle point as the action on $\mathcal{D}^{\mathbf{P}}_\Omega$ is via multiplying with the character $\chi_\Omega : \mathbf{L}_{\mathbf{P}}(\mathbb{Z}_p) \to \mathcal{O}_\Omega$, but this can be lifted to $\chi_\Omega : \mathbf{L}_{\mathbf{B}}(\mathbb{Z}_p) \to \mathcal{O}_\Omega$ which is the $\mathrm{Iw}^\beta_{\mathbf{B}}$-action. It then remains to the specialisation. We then show that the left hand side equals the right hand side by unpacking the definitions and calculating:

$$\begin{aligned}\mathrm{sp}_{\underline{\lambda}}\left(\mathrm{Vol}(L^\beta_{\mathbf{P}}) \sum_{i=1}^{r} \frac{h_i * (\kappa^\beta_\Omega \circ \mathcal{F}(h_i))(\chi_{[p]})}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^\beta_{\mathbf{P}} h_i^{-1}|}\right) &= \mathrm{Vol}(L^\beta_{\mathbf{P}}) \sum_{i=1}^{r} \frac{\kappa^\beta_{\underline{\lambda}} \circ \mathrm{sp}_{\underline{\lambda}}(\mathcal{F}(h_i)(h_i^{-1} * \chi_{[p]}))}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^\beta_{\mathbf{P}} h_i^{-1}|} \\ &= \mathrm{Vol}(L^\beta_{\mathbf{P}}) \sum_{i=1}^{r} \frac{\kappa^\beta_{\underline{\lambda}} \circ \mathrm{sp}_{\underline{\lambda}}(\mathcal{F}(h_i)(\eta_{[p]}(\det h_{i,2}^{-1})\chi_{[p]}(h_{i,2}h_{i,1}^{-1})\chi_{[p]}))}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^\beta_{\mathbf{P}} h_i^{-1}|} \\ &= \mathrm{Vol}(L^\beta_{\mathbf{P}}) \sum_{i=1}^{r} \frac{\eta_{[p]}(\det h_{i,2}^{-1})\chi_{[p]}(\det(h_2 h_1^{-1}))\kappa^p_{\underline{\lambda},\underline{j}_\chi} \circ \mathrm{sp}_{\underline{\lambda}}(\mathcal{F}(h_i))}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^\beta_{\mathbf{P}} h_i^{-1}|}\end{aligned}$$

where we use theorem 3.7 in the first and last equality. □

We further define translation map

$$R^{\mathbf{P}}_\beta : M_p(\mathcal{D}^{\mathbf{P}}_\Omega, K^{\mathbf{P}}_f) \to M_p(\mathcal{D}^{\mathbf{P},\beta}_\Omega, K^{\mathbf{P},\beta}_f); \mathcal{F} \mapsto \left(g \mapsto \prod_{\mathfrak{p}|p} u_{\mathfrak{p}}^{-1} t^\beta_{\mathfrak{p},\mathbf{P}} * \mathcal{F}\left(g \prod_{\mathfrak{p}|p} u_{\mathfrak{p}}^{-1} t^\beta_{\mathfrak{p},\mathbf{P}}\right)\right) \tag{4.3}$$

With this, we will now translate the algebraic modular forms such that they permit p-adic interpolation.

Lemma 4.2. *Let $\mathfrak{p}|p$ be a prime ideal, $\mathbf{P}$ be a relevant parabolic, $K^{\mathbf{P}}_f$ an open compact subgroup, $\Omega \subseteq \mathscr{W}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}_{K,0}(L)$ an affinoid and $\beta \in \mathbb{Z}^{\mathfrak{p}|p}_{>0}$. Then we have for $\mathcal{F} \in M_p(\mathcal{D}^{\mathbf{P}}_\Omega, K^{\mathbf{P}}_f)$ that*

$$Ev^{\beta'}_\Omega(R^{\mathbf{P}}_{\beta'}(\mathcal{F})) = \delta_{\mathbf{B}}(t^{-1}_{\mathbf{P},\mathfrak{p}}) Ev^{\beta'}_\Omega(R^{\mathbf{P}}_\beta(U_{\mathbf{P},\mathfrak{p}}\mathcal{F})) \tag{4.4}$$

where $\beta'_{\mathfrak{p}} = \beta_{\mathfrak{p}} + 1$ and $\beta'_{\mathfrak{q}} = \beta_{\mathfrak{q}}$ for $\mathfrak{q} \neq \mathfrak{p}$. Furthermore, if $U^\circ_{\mathbf{P},p}\mathcal{F} = \alpha^\circ_{\mathbf{P},p}\mathcal{F}$, there we have independent of the choice of β a well-defined map

$$\mu(\mathcal{F}) := \delta_{\mathbf{B}}(t^{-\beta}_{\mathbf{P},p})(\alpha^\circ_{\mathbf{P},p})^{-\beta} Ev^\beta_\Omega(R^{\mathbf{P}}_\beta(\mathcal{F})). \tag{4.5}$$

Proof. To apply the Hecke operator, we also expand the double coset as

$$\mathrm{Iw}_{\mathbf{P},\mathfrak{p}}\, t_{\mathbf{P},\mathfrak{p}}\, \mathrm{Iw}_{\mathbf{P},\mathfrak{p}} = \bigsqcup_{j \in L} \delta_j t_{\mathbf{P},\mathfrak{p}}\, \mathrm{Iw}_{\mathbf{P},\mathfrak{p}} \tag{4.6}$$

with representatives $\delta_j t_{\mathbf{P},\mathfrak{p}}$ where $j \in L := L^{\beta}_{\mathbf{P},p}/L^{\beta'}_{\mathbf{P},p} \leftrightarrow \delta_j = (u^{-1}t^{\beta}_{\mathbf{P},p})^{-1} j (u^{-1}t^{\beta}_{\mathbf{P},p}) \in \mathrm{Iw}_{\mathbf{P}}/(\mathrm{Iw}_{\mathbf{P}} \cap t_{\mathbf{P},\mathfrak{p}} \mathrm{Iw}_{\mathbf{P}} t^{-1}_{\mathbf{P},\mathfrak{p}})$. We then prove the main statement by direct calculation.

$$\begin{aligned}
\mathrm{Ev}^{\beta'}_{\Omega}(R^{\mathbf{P}}_{\beta}(U^{\circ}_{\mathbf{P},\mathfrak{p}}\mathcal{F})) &= \mathrm{Vol}(L^{\beta'}_{\mathbf{P}}) \sum_{i=1}^{r} \frac{h_i * \kappa^{\beta'}_{\Omega}\left(R^{\mathbf{P}}_{\beta}(U^{\circ}_{\mathbf{P},\mathfrak{p}}\mathcal{F})(h_i)\right)}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^{\beta'}_{\mathbf{P}} h_i^{-1}|} \\
&= \mathrm{Vol}(L^{\beta'}_{\mathbf{P}}) \sum_{i=1}^{r} \frac{h_i * \kappa^{\beta'}_{\Omega}\left(u^{-1}t^{\beta}_{\mathbf{P},p} \sum_{j\in L}(\delta_j t_{\mathbf{P},\mathfrak{p}})\mathcal{F}\left(h_i u^{-1} t^{\beta}_{\mathbf{P},p}(\delta_j t_{\mathbf{P},\mathfrak{p}})\right)\right)}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^{\beta'}_{\mathbf{P}} h_i^{-1}|} \\
&= \mathrm{Vol}(L^{\beta'}_{\mathbf{P}}) \sum_{i=1}^{r} \frac{h_i * \kappa^{\beta}_{\Omega}\left(\sum_{j\in L}(j u^{-1} t^{\beta'}_{\mathbf{P},p})\mathcal{F}\left(h_i j u^{-1} t^{\beta'}_{\mathbf{P},p}\right)\right)}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^{\beta'}_{\mathbf{P}} h_i^{-1}|} \\
&= \mathrm{Vol}(L^{\beta'}_{\mathbf{P}}) \sum_{i=1}^{r} \sum_{j\in L} \frac{(h_i j) * \kappa^{\beta}_{\Omega}\left((u^{-1}t^{\beta'}_{\mathbf{P},p})\mathcal{F}\left(h_i j u^{-1} t^{\beta'}_{\mathbf{P},p}\right)\right)}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^{\beta'}_{\mathbf{P}} h_i^{-1}|} \\
&= \mathrm{Vol}(L^{\beta'}_{\mathbf{P}})[L^{\beta_{\mathfrak{p}}}_{\mathbf{P},\mathfrak{p}} : L^{\beta_{\mathfrak{p}}+1}_{\mathbf{P},\mathfrak{p}}] \sum_{i=1}^{r} \frac{h_i * \kappa^{\beta'}_{\Omega}\left(R^{\mathbf{P}}_{\beta'}\mathcal{F}(h_i)\right)}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^{\beta'}_{\mathbf{P}} h_i^{-1}|} = \delta_{\mathbf{B}}(t^{-1}_{\mathbf{P},\mathfrak{p}}) \mathrm{Ev}^{\beta'}_{\Omega}(R^{\mathbf{P}}_{\beta'}(\mathcal{F}))
\end{aligned}$$

In the above, the first equality is the definition, the second equality is unpacking the translation map and $U^{\circ}_{\mathfrak{p},\mathbf{P}}$, the third equality uses $u^{-1}t^{\beta}_{\mathbf{P},p}\delta_j t_{\mathfrak{p},\mathbf{P}} = j u^{-1} t^{\beta'}_{\mathbf{P},p}$, the fourth equality arises from the $\mathrm{Iw}^{\beta}_{\mathbf{P}}$-equivariance and the fifth notes that the j just permutes the h_i with $|L|$ amount of entries. It immediately follows that μ is well-defined considering that $\mathcal{F}$ is a $U^{\circ}_{\mathbf{P},p}$-eigenvector with eigenvalue α°_p. □

We will see that this map will yield us our desired distribution for interpolation. It remains to calculate its p-adic growth.

Proposition 4.3. *Let $\mathbf{P}$ be a relevant parabolic, $K^{\mathbf{P}}_f$ an open compact subgroup and $\Omega \subseteq \mathscr{W}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}_{K,0}(L)$ an affinoid and $\mathcal{F} \in M(\mathcal{D}^{\mathbf{P}}_{\Omega}, K^{\mathbf{P}}_f)$ such that $U^{\circ}_{\mathbf{P},p}\mathcal{F} = \alpha^{\circ}_{\mathbf{P},p}\mathcal{F}$. Then μ has growth $\underline{h} = (v_p(\alpha^{\circ}_{\mathbf{P},\mathfrak{p}}))_{\mathfrak{p}|p}$.*

Proof. To prove this, we need to show that there exists $C \in \mathbb{R}_{>0}$ such that for all $\underline{m} \in \mathbb{Z}^{\mathfrak{p}|p}_{\geq 1}$, we have:

$$||\mu||_{\underline{m}} \leq C p^{\underline{hm}}, \tag{4.7}$$

To prove this, we will show the following set of inequalities.

$$\begin{aligned}
||\mu||_{\underline{m}} &= ||\delta_{\mathbf{B}}(t^{-\underline{m}}_{\mathbf{P},p})(\alpha^{\circ}_p)^{-\underline{m}} \mathrm{Ev}_{\Omega}(R^{\mathbf{P}}_{\underline{m}}(\mathcal{F}))||_{\underline{m}} \\
&= |\delta_{\mathbf{B}}(t^{-\underline{m}}_{\mathbf{P},p})(\alpha^{\circ}_p)^{-\underline{m}} \mathrm{Vol}(L^{\underline{m}}_{\mathbf{P}})| \left|\left| \sum_{i=1}^{r} \frac{h_i * \kappa_{\Omega}\left(R^{\mathbf{P}}_{\beta}\mathcal{F}(h_i)\right)}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^{\beta}_{\mathbf{P}} h_i^{-1}|} \right|\right|_{\underline{m}} \\
&\leq p^{\underline{mh}} |\mathrm{Vol}(L^{1}_{\mathbf{P}})| \max_i \frac{\left|\left| h_i * \kappa_{\Omega}\left(R^{\mathbf{P}}_{\beta}\mathcal{F}(h_i)\right)\right|\right|_{\underline{m}}}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^{\underline{m}}_{\mathbf{P}} h_i^{-1}|} \\
&\leq p^{\underline{mh}} C \max_i \frac{1}{|\mathbf{H}(\mathbb{Q}) \cap h_i L^{\underline{m}}_{\mathbf{P}} h_i^{-1}|} \leq C' p^{mh}.
\end{aligned}$$

The first and second equality are just unpacking the definitions. The first inequality arises from the ultra-metric inequality and that $\delta_{\mathbf{B}}(t^{-\underline{m}}_{\mathbf{P},p})\,\mathrm{Vol}(L^{\underline{m}}_{\mathbf{P}}) = \mathrm{Vol}(L^{1}_{\mathbf{P}})$. The last inequality follows from $1/|\mathbf{H}(\mathbb{Q}) \cap h_i L^{\underline{m}}_{\mathbf{P}} h_i^{-1}| \leq 1$. To prove the second inequality, it remains to bound the expression

$$||h_i * \kappa_{\Omega}\left(R^{\mathbf{P}}_{\underline{m}}\mathcal{F}(h_i)\right)(g)||_{\underline{m}} = \sup_{||g||_{\underline{m}} \leq 1} |h_i * \kappa_{\Omega}\left(R^{\mathbf{P}}_{\underline{m}}\mathcal{F}(h_i)\right)(g)|. \tag{4.8}$$

To do this, we begin with rewriting the expression

$$\begin{aligned} h_i * \kappa_\Omega \left(R^{\mathbf{P}}_{\underline{m}} \mathcal{F}(h_i) \right)(g) &= ut^{\underline{m}}_{\mathbf{P},p} \mathcal{F}(h_i ut^{\underline{m}}_{\mathbf{P},p})(v_\Omega(h_i^{-1} * g)) \\ &= \mathcal{F}(h_i ut^{\underline{m}}_{\mathbf{P},p})(ut^{\underline{m}}_{\mathbf{P},p} * v_\Omega(h_i^{-1} * g)) \end{aligned}$$

where $g \in \mathcal{A}_{\underline{m}}(\Gamma^{\mathrm{ac}}_{p^\infty \mathfrak{R}}, L)$ and $||g||_{\underline{m}} \leq 1$. We observe that h_i^{-1} acts isometrically so $h_i^{-1} * g \in \mathcal{A}_{\underline{m}}(\Gamma^{\mathrm{ac}}_{p^\infty \mathfrak{R}}, L)$ and $||h_i^{-1} * g||_{\underline{m}} \leq 1$. Then we note that v_Ω preserves $\underline{m}$-analyticity, so $v_\Omega(h_i^{-1} * g) \in \mathcal{A}_{\underline{m}}(\Gamma^{\mathrm{ac}}_{p^\infty \mathfrak{R}}, L)$ and $||v_\Omega(h_i^{-1} * g)||_{\underline{m}} \leq 1$. Lastly, $(ut^{\underline{m}}_{\mathbf{P},p})^{-1}$ lowers the depth so that $(ut^{\underline{m}}_{\mathbf{P},p})^{-1} v_\Omega(h_i^{-1} * g) \in \mathcal{A}_1(\Gamma^{\mathrm{ac}}_{p^\infty \mathfrak{R}}, L)$ and $||(ut^{\underline{m}}_{\mathbf{P},p})^{-1} v_\Omega(h_i^{-1} * g)||_1 \leq 1$. It then remains to show that $\mathcal{F}(h_i ut^{\underline{m}}_{\mathbf{P},p})|_{\mathcal{A}_1(\Gamma^{\mathrm{ac}}_{p^\infty \mathfrak{R}}, L)}$ is bounded for $||g||_1 \leq 1$. We note that h_i is a representative for $[H]/L^{\underline{m}}_{\mathbf{P}}$. Hence $h_i ut^{\underline{m}}_{\mathbf{P},p}$ is a representative for $[H]/L^{\underline{1}}_{\mathbf{P}}$. Consequently, there exists $\gamma \in \mathbf{H}(\mathbb{Q})$, $k \in L^{\underline{1}}_{\mathbf{P}}$ and some representative $h' \in [H]/L^{\underline{m}}_{\mathbf{P}}$ such that $h_i ut^{\underline{m}}_{\mathbf{P},p} = \gamma h' k$. We then have:

$$||\mathcal{F}_1(h_i ut^{\underline{m}}_{\mathbf{P},p})||_1 = ||\mathcal{F}_1(\gamma h' k)||_1 = ||k_p^{-1} \mathcal{F}_1(h')||_1 = ||\mathcal{F}_1(h')||_1 \leq C := \max_j ||\mathcal{F}_1(h_j)||_1 \tag{4.9}$$

where h_j are the finite representatives of $[H]/L^{\underline{1}}_{\mathbf{P}}$ and $\mathcal{F}_1 := \mathcal{F}|_{\mathcal{A}_1(\Gamma^{\mathrm{ac}}_{p^\infty \mathfrak{R}}, L)}$. Consequently it follows that

$$\begin{aligned} |h_i * \kappa_\Omega \left(R^{\mathbf{P}}_{\underline{m}} \mathcal{F}(h_i) \right)(g)| &= |\mathcal{F}(h_i ut^{\underline{m}}_{\mathbf{P},p})(ut^{\underline{m}}_{\mathbf{P},p} * v_\Omega(h_i^{-1} * g))| \\ &= |\mathcal{F}(h_i ut^{\underline{m}}_{\mathbf{P},p})|_{\mathcal{A}_1(\Gamma^{\mathrm{ac}}_{p^\infty \mathfrak{R}}, L)}(ut^{\underline{m}}_{\mathbf{P},p} * v_\Omega(h_i^{-1} * g))| \\ &\leq ||\mathcal{F}(h_i ut^{\underline{m}}_{\mathbf{P},p})|_{\mathcal{A}_1(\Gamma^{\mathrm{ac}}_{p^\infty \mathfrak{R}}, L)}||_1 \leq C \end{aligned}$$

where the first inequality comes from $||\mu||_{\underline{m}} = \sup_{||g||_{\underline{m}} \leq 1} |\mu(f)|$. □

4.2 Lifts to distributions

It now remains to apply the evaluation maps to automorphic vectors $\varphi \in \pi$.

Theorem 4.4. *Suppose* $\mathbf{P}$ *be a relevant parabolic,* π_E *an E-model of a RACAR of* $\mathbf{G}$ *and* $\tilde{\pi}_E$ *its* $\mathbf{P}$*-non-critical* $\mathbf{P}$*-refinement and* $v_p(\alpha^\circ_{\mathbf{P},p}) \neq \infty$. *Then for every* $\varphi_f \in \tilde{\pi}^{K_f}_{E,f}$, *there exists a distribution*

$$\mu_{\tilde{\pi}}(\varphi) \in \mathcal{D}^h(\Gamma^{ac}_{p^\infty \mathfrak{R}}, L) \tag{4.10}$$

of growth at most $\underline{h} = (v_p(\alpha^\circ_{\mathbf{P},\mathfrak{p}}))_{\mathfrak{p}|p}$ *such that for every anti-cyclotomic Hecke characters* $\chi \in \mathrm{Crit}(\underline{\lambda}_\pi)[p^\beta \mathfrak{R}$ *with* $\beta \in \mathbb{Z}^{\mathfrak{p}|p}_{\geq 1}$, *we have that*

$$\mu_{\tilde{\pi}}(\varphi)(\chi_{[p]}) = \frac{\delta_{\mathbf{B}}(t^{-\beta}_{\mathbf{P},p})}{\alpha^\beta_p} \mathscr{P}^\infty_{\mathbf{H},\chi}((ut^\beta_{\mathbf{P},p} \varphi_f) \otimes v^\vee_{\underline{\lambda}_\pi, \underline{j}_\chi}). \tag{4.11}$$

Proof. By lemma 2.2, we have $F_\varphi \in M(V^\vee_{\underline{\lambda}_\pi}, K_f)^{\leq h}_{\tilde{\pi}}$ for $h \in \mathbb{Q}_{>0}$ sufficiently large. Because $\tilde{\pi}$ being non-$\mathbf{P}$-critical, we have that:

$$f_\varphi(g) := g_p^{-1} F_\varphi(g) \in M_p(V^\vee_{\underline{\lambda}}(L), K_f)^{\leq h}_{\tilde{\pi}} \cong M_p(\mathcal{D}^{\mathbf{P}}_{\underline{\lambda}}, K_f)^{\leq h}_{\tilde{\pi}}. \tag{4.12}$$

Hence, we view f_φ as an element of $M_p(\mathcal{D}^{\mathbf{P}}_{\underline{\lambda}}, K_f)_{\tilde{\pi}}$. We then get claimed distribution by lemma 4.2 as $\mu_{\tilde{\pi}}(\varphi) := \mu_{\underline{\lambda}_{\tilde{\pi}}}(f_\varphi) \in \mathcal{D}^{\underline{h}}(\Gamma^{\mathrm{ac}}_{p^\infty \mathfrak{R}}, L)$. Applying lemma 4.1, we get that this distribution descends to the classical evaluation map for the desired anti-cyclotomic Hecke characters $\chi \in \mathrm{Crit}(\underline{\lambda}_{\tilde{\pi}})[p^\beta \mathfrak{R}]$. Using proposition 2.3, we then get that these classical evaluation maps correspond to the period integral at $R^{\mathbf{P}}_\beta f_\varphi = R^{\mathbf{P}}_\beta \Omega_{\tilde{\pi}} \mathcal{F}_\varphi$. Lastly, the growth property comes from proposition 4.3. □

Lemma 4.5. *Suppose the assumptions of theorem 4.4 and let $\underline{\lambda}_{\tilde{\pi}} \in \Omega \subseteq \mathscr{W}_{K,0}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}(L)$ be a connected affinoid slope-adapted to some $h \in \mathbb{Q}_{>0}$. Let $f_1, .., f_m \in M_p(V^\vee_{\underline{\lambda}_{\tilde{\pi}}}(L), K_f^{\mathbf{P}})$ be a basis. Then there exists a basis $f_{1,\Omega}, .., f_{m,\Omega} \in M(\mathcal{D}^{\mathbf{P}}_{\Omega}, K_f^{\mathbf{P}})^{\leq h}_{\tilde{\pi}}$ such that:*

1. $M(\mathcal{D}^{\mathbf{P}}_{\Omega}, K_f^{\mathbf{P}})^{\leq h}_{\tilde{\pi}} = \oplus_{i=1}^m \mathcal{O}_\Omega f_{i,\Omega}$, *and*
2. $\iota_{\mathbf{G}/\mathbf{P}} \circ sp_{\underline{\lambda}}(f_{i,\Omega}) = f_i$.

Proof. We lift it in two steps. First, from $\tilde{\pi}$ being non-$\mathbf{P}$-critical, we have that $f_1, .., f_m \in M_p(V^\vee_{\underline{\lambda}}, K_f^{\mathbf{P}})_{\tilde{\pi}} \cong M_p(\mathcal{D}^{\mathbf{P}}_{\underline{\lambda}}, K_f^{\mathbf{P}})_{\tilde{\pi}}$. For the second step, recall that $\mathcal{D}^{\mathbf{P}}_{\Omega} \otimes_{\mathcal{O}_\Omega} \mathcal{O}_\Omega/\mathfrak{m}_{\underline{\lambda}} = \mathcal{D}^{\mathbf{P}}_{\underline{\lambda}}$. Hence we have

$$\frac{M_p(\mathcal{D}^{\mathbf{P}}_{\Omega}, K_f^{\mathbf{P}})}{\mathfrak{m}_{\underline{\lambda}} M_p(\mathcal{D}^{\mathbf{P}}_{\Omega}, K_f)} \cong M_p(\mathcal{D}^{\mathbf{P}}_{\Omega}, K_f^{\mathbf{P}}) \otimes_{\mathcal{O}_\Omega} \frac{\mathcal{O}_\Omega}{\mathfrak{m}_{\underline{\lambda}}} \cong M_p\left(\mathcal{D}^{\mathbf{P}}_{\Omega} \otimes_{\mathcal{O}_\Omega} \frac{\mathcal{O}_\Omega}{\mathfrak{m}_{\underline{\lambda}}}, K_f^{\mathbf{P}}\right) = M_p(\mathcal{D}^{\mathbf{P}}_{\underline{\lambda}}, K_f^{\mathbf{P}}) \tag{4.13}$$

where the middle isomorphism follows from

$$M_p(\mathcal{D}^{\mathbf{P}}_{\Omega}, K_f^{\mathbf{P}}) \otimes_{\mathcal{O}_\Omega} \frac{\mathcal{O}_\Omega}{\mathfrak{m}_{\underline{\lambda}}} \cong \left(\bigoplus_i (\mathcal{D}^{\mathbf{P}}_{\Omega})^{\Gamma_i}\right) \otimes \frac{\mathcal{O}_\Omega}{\mathfrak{m}_{\underline{\lambda}}} \cong \bigoplus_i \left(\mathcal{D}^{\mathbf{P}}_{\Omega} \otimes \frac{\mathcal{O}_\Omega}{\mathfrak{m}_{\underline{\lambda}}}\right)^{\Gamma_i} \cong M_p\left(\mathcal{D}^{\mathbf{P}}_{\Omega} \otimes_{\mathcal{O}_\Omega} \frac{\mathcal{O}_\Omega}{\mathfrak{m}_{\underline{\lambda}}}, K_f^{\mathbf{P}}\right) \tag{4.14}$$

where we use the ideas of [Gro99, Proposition 4.5] and Γ_i are the stabilisers as defined the reference. However, $\mathcal{D}^{\mathbf{P}}_{\Omega}$ is infinite dimensional, we use the finite slope condition to ensure we have a finite rank object which is needed for Nakayama's lemma. Hence we have from equation 4.13 that

$$\frac{M_p(\mathcal{D}^{\mathbf{P}}_{\Omega}, K_f^{\mathbf{P}})^{\leq h}_{\tilde{\pi}}}{\mathfrak{m}_{\underline{\lambda}} M_p(\mathcal{D}^{\mathbf{P}}_{\Omega}, K_f)^{\leq h}_{\tilde{\pi}}} \cong M_p(\mathcal{D}^{\mathbf{P}}_{\underline{\lambda}}, K_f^{\mathbf{P}})^{\leq h}_{\tilde{\pi}}. \tag{4.15}$$

Consequently, we know it is generated by elements $f_{1,\Omega}, .., f_{m,\Omega}$ that specialises to $f_1, .., f_m$. Lastly, we need to show it is freely generated. Let $f \in M_p(\mathcal{D}^{\mathbf{P}}_{\Omega}, K_f)$, assume $f \neq 0$ and let $x \in \mathcal{O}_\Omega$ such that $xf = 0$. Then there exist $g \in G(\mathbb{Q})$ such that $\mu = f(g) \neq 0 \in \mathcal{D}^{\mathbf{P}}_{\Omega}$ and $x\mu = 0$. As $\mathcal{O}_\Omega$ is an integral domain, we have that $x = 0$ and hence $M_p(\mathcal{D}^{\mathbf{P}}_{\Omega}, K_f^{\mathbf{P}})^{\leq 0}_{\tilde{\pi}}$ is torsion free. □

From the remaining theorems, we assume K_v are hyper special maximal compact for π_v ramified, S is a finite set containing these v, we have global multiplicity $m(\pi) = 1$ which is true if $\mathrm{BC}_{F/F^+}(\pi)$ is cuspidal and $K_f^{\mathbf{P}} = K^p \mathrm{Iw}_{\mathbf{P}}$ and $K = K_f \mathbf{G}(\mathbb{R})$. We further assume that for all classical E_v-model $v \in \Omega$, $\pi^{K_v}_{v,v,E_v}$ exists.

Theorem 4.6. *Suppose the assumptions of theorem 4.4 are valid. Then for every $h \geq v_p(\alpha_p)$, there exists an h-adapted neighbourhood $\Omega \subseteq \mathscr{W}_{K,0}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}(L)$ of $\underline{\lambda}_\pi$ and family $\tilde{\pi} \in \mathscr{C} \subseteq \mathscr{E}^{\mathbf{P}}$ such that:*

$$w : \mathscr{C} \cong \Omega \text{ and } w(\tilde{\pi}) = \underline{\lambda}_{\tilde{\pi}} \tag{4.16}$$

and it satisfies the following properties:

1. *We have $\dim \mathscr{C} = \dim \Omega = \dim \mathscr{W}_{K,0}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}(L)$, and*
2. *The set $\mathscr{C}_{cl}$ of classical points is dense in $\mathscr{C}$.*

Proof. We begin with the main claim and recall from lemma 2.1 that

$$M_p(V_{\underline{\lambda}_\pi}, K_f^{\mathbf{P}})[\pi] \cong L \otimes \pi_{f,E}^{K_f^{\mathbf{P}}} \otimes \mathcal{M} \tag{4.17}$$

where $\mathcal{M} = \mathrm{Hom}_{\mathbf{G}(\mathbb{A}_{\mathbb{Q},f})}(\pi_{E,f}, M(V^\vee_{\underline{\lambda}}(E), K_f)[\pi])$. By the global multiplicity hypothesis, we have the multiplicity space is 1. As $\tilde{\pi}$ is non-$\bar{\mathbf{P}}$-critical, we thus have

$$M_p(\mathcal{D}^{\mathbf{P}}_{\underline{\lambda}_\pi}, K_f^{\mathbf{P}})_{\tilde{\pi}} \cong M_p(V_{\underline{\lambda}_\pi}, K_f^{\mathbf{P}})_{\tilde{\pi}} \cong \bigotimes_{v \in S} \pi_{v,E}^{K_v} \otimes L. \tag{4.18}$$

Let $\Omega \subseteq \mathscr{W}_{K,0}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}(L)$ be a neighbourhood of $\underline{\lambda}_{\tilde{\pi}}$ which exists by [BGR84, Chapter 9, Definition 4.]. We then shrink Ω such that it is h-slope adapted and observe from lemma 4.5 that

$$M_p(\mathcal{D}_\Omega^{\mathbf{P}}, K_f^{\mathbf{P}})_{\tilde{\pi}}^{\leq h} \cong \bigotimes_{v\in S} \pi_{v,E}^{K_v} \otimes \mathcal{O}_\Omega. \tag{4.19}$$

With this in mind, we recall the local eigenvariety chart

$$\mathscr{E}_{\Omega,\leq h}^{\mathbf{P}} = \mathrm{Sp}(\mathbb{T}_{\Omega,\leq h}^{\mathbf{P}}) \text{ with } \mathbb{T}_{\Omega,\leq h}^{\mathbf{P}} = \mathrm{Im}(\mathcal{H}_{\mathbf{P}}^S \otimes \mathcal{O}_\Omega \to \mathrm{End}_{\mathcal{O}_\Omega} M_p(\mathcal{D}_\Omega^{\mathbf{P}}, K_f^{\mathbf{P}})_{\tilde{\pi}}^{\leq h}). \tag{4.20}$$

We will want to show that $\mathbb{T}_{\Omega,\leq h}^{\mathbf{P}} \cong \mathcal{O}_\Omega$. Because $M(\mathcal{D}_\Omega, K_f)_{\tilde{\pi}}^{\leq h} \cong \mathcal{O}_\Omega^d$ where $d = \dim \tilde{\pi}_{f,E}^{K_f^{\mathbf{P}}}$, we have that $\mathrm{End}_{\mathcal{O}_\Omega} M(\mathcal{D}_\Omega, K_f)_{\tilde{\pi}}^{\leq h} \cong M_d(\mathcal{O}_\Omega)$. If S is empty, then $d = 1$ and we get the desired result automatically. Assume S is non-empty. To control the bad primes, we introduce an auxiliary ring of Hecke operators

$$\mathcal{H}_{bad} := \bigotimes_{v\in S} C_c^\infty(K_v \setminus \mathbf{G}(F_v^+)/K_v, E). \tag{4.21}$$

Via the surjection $\mathcal{H}_{bad} \twoheadrightarrow \mathrm{End}_E M(\mathcal{D}_{\underline{\lambda}_\pi}^{\mathbf{P}}(E), K_f^{\mathbf{P}})_{\tilde{\pi}}^{\leq h} \cong M_d(E)$ of [GH24, Theorem 4.5.2 and Proposition 5.3.3], we have that we have that $\mathrm{Im}\,\mathcal{H}_{bad} = \mathrm{End}_E M(\mathcal{D}_{\underline{\lambda}_\pi}^{\mathbf{P}}(E), K_f^{\mathbf{P}})_{\tilde{\pi}}^{\leq h}$. Up to shrinking Ω, it follows that

$$\mathrm{Im}(\mathcal{H}_{bad}) \otimes \mathcal{O}_\Omega \cong \mathrm{End}_{\mathcal{O}_\Omega} M_p(\mathcal{D}_\Omega^{\mathbf{P}}, K_f^{\mathbf{P}})_{\tilde{\pi}}^{\leq h}. \tag{4.22}$$

However, for all $T \in \mathcal{H}_{\mathbf{P}}^S$ and $T' \in \mathcal{H}_{bad}$ we have that $TT' = T'T$. Hence follows that our Hecke algebra lie in its commutator, that is

$$\mathrm{Im}(\mathcal{H}_{\mathbf{P}}^S) \otimes \mathcal{O}_\Omega \subseteq \{T \in \mathrm{End}_{\mathcal{O}_\Omega} M_p(\mathcal{D}_\Omega^{\mathbf{P}}, K_f^{\mathbf{P}})_{\tilde{\pi}}^{\leq h} : Ta = aT \ \forall a \in \mathrm{Im}(\mathcal{H}_{bad}) \otimes \mathcal{O}_\Omega\}. \tag{4.23}$$

Furthermore, equation 4.22 yield that the commutator is the centraliser

$$Z(\mathrm{End}_{\mathcal{O}_\Omega} M_p(\mathcal{D}_\Omega^{\mathbf{P}}, K_f^{\mathbf{P}})_{\tilde{\pi}}^{\leq h}) \cong Z(M_d(\mathcal{O}_\Omega)) = \mathcal{O}_\Omega. \tag{4.24}$$

Hence we get the desired $\mathbb{T}_{\Omega,\leq h}^{\mathbf{P}} \cong \mathcal{O}_\Omega$. It follows that $w : \mathscr{E}_{\Omega,\leq h}^{\mathbf{P}} \cong \Omega$. By construction, we have that $\tilde{\pi} \in \mathscr{E}_{\Omega,\leq h}^{\mathbf{P}}$ and

$$w : \mathscr{E}_{\Omega,\leq h}^{\mathbf{P}} \to \Omega \text{ with } w(\tilde{\pi}) = \underline{\lambda}_{\tilde{\pi}}. \tag{4.25}$$

We now move on to the properties.

1. Because Ω is an open set in $\Omega \subseteq \mathscr{W}_{K,0}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}(L)$, it follows that we have:

$$\dim \Omega = \dim \mathscr{W}_{K,0}^{\mathrm{Res}_{F^+/\mathbb{Q}}\mathbf{P}}(L). \tag{4.26}$$

   The last statement follows from $\mathscr{E}_{\Omega,\leq h} \cong \Omega$, so we have that $\dim \Omega = \dim \mathscr{E}_{\Omega,\leq h}$.

2. This property now follows from [SW21, Proposition 5.15].

□

Theorem 4.7. *Suppose we have a family $\mathscr{C}$ and neighbourhood Ω as in theorem 4.6. Then there exists a unique distribution*

$$\mu_\Omega(\varphi) \in \mathcal{D}(\Gamma_{p^\infty\mathfrak{R}}^{ac}, \mathcal{O}_\Omega) \tag{4.27}$$

such that for every classical $\upsilon \in \Omega$, there exist p-adic periods $c_\upsilon \in L^\times$ and pure vector $\varphi_\upsilon \in \tilde{\pi}_{\upsilon,f,E_\upsilon}^{K_f^{\mathbf{P}}}$ such that we have:

$$sp_\upsilon(\mu_\Omega(\varphi)) = c_\upsilon \mu_{\tilde{\pi}}(\varphi_\upsilon). \tag{4.28}$$

where $\mu_{\tilde{\pi}}$ is from theorem 4.4.

Proof. We will consider two cases: $S = \varnothing$ and $S \neq \varnothing$, and in both cases we will find a suitable f_Ω and use lemma 4.2 to define

$$\mu_\Omega(\varphi) = \mu(f_\Omega). \tag{4.29}$$

In either case, the key idea is to find a family and compatible rational lines at the classical specialisations. We begin with the case $S = \varnothing$. Then we have from the properties of $\mathscr{C}$ and Ω with respect to theorem 4.6, we can apply lemma 4.5 to get an element

$$\mathcal{O}_\Omega f_\Omega = M_p(\mathcal{D}_\Omega^{\mathbf{P}}, K_f^{\mathbf{P}})_{\tilde{\pi}}^{\leq h} \tag{4.30}$$

for an $h \geq v_p(\alpha_p)$ such that $\mathrm{sp}_{\underline{\lambda}_\pi}(f_\Omega) = f_{\underline{\lambda}_\pi} \in M_p(\mathcal{D}_{\underline{\lambda}_\pi}^{\mathbf{P}}, K_f^{\mathbf{P}})$. We then note that for all classical $\upsilon \in \Omega_{\mathrm{cl}}$, we have that

$$\mathrm{sp}_\upsilon(f_\Omega) \in Lf_\upsilon = M_p(\mathcal{D}_\upsilon^{\mathbf{P}}, K_f^{\mathbf{P}})_{\tilde{\pi}_\upsilon}^{\leq h}. \tag{4.31}$$

Without loss of generality, we can assume $g_p f_\upsilon \in M(V_\upsilon^\vee(E_\upsilon), K_f^{\mathbf{P}})_{\tilde{\pi}_\upsilon}^{\leq h}$ for a number field E_υ embedding into L. Hence there exists $c_\upsilon \in L$ such that $\mathrm{sp}_\upsilon(f_\Omega) = c_\upsilon f_\upsilon$ and due to lemma 2.1 and 2.2 there exists a corresponding $\varphi_{f,\upsilon} \in \tilde{\pi}_{\upsilon,E_\upsilon}$. We now assume $S \neq \varnothing$ and begin with studying the structure of $M_p(\mathcal{D}_\upsilon^{\mathbf{P}}, K_f^{\mathbf{P}})_{\tilde{\pi}_\upsilon}^{\leq h}$ for classical $\upsilon \in \Omega$. From lemma 4.5, we immediately have that

$$\dim M_p(\mathcal{D}_\upsilon^{\mathbf{P}}, K_f^{\mathbf{P}})_{\tilde{\pi}_\upsilon}^{\leq h} = \mathrm{rank} M_p(\mathcal{D}_\Omega^{\mathbf{P}}, K_f^{\mathbf{P}})_{\tilde{\pi}_\upsilon}^{\leq h} = \dim M_p(\mathcal{D}_{\underline{\lambda}}^{\mathbf{P}}, K_f^{\mathbf{P}})_{\tilde{\pi}_{\underline{\lambda}}}^{\leq h}. \tag{4.32}$$

and from lemma 2.1 and 2.2, we have that for a sufficiently large number field $E_\upsilon \subseteq L$ that

$$M_p(\mathcal{D}_\upsilon^{\mathbf{P}}, K_f^{\mathbf{P}})_{\tilde{\pi}_\upsilon}^{\leq h} \cong L^{m(\pi_\upsilon)} \otimes \pi_{\upsilon,f,E_\upsilon}^{K_f^{\mathbf{P}}}. \tag{4.33}$$

We now claim that for each $\upsilon \in S$ we have $\dim \pi_{\upsilon,\upsilon,E_\upsilon}^{K_f^{\mathbf{P}}} = \dim \pi_{\upsilon,E}^{K_f^{\mathbf{P}}}$, and this immediately implies $m(\pi_\upsilon) = 1$. Similarly to above, we have locally $\mathrm{Im}(\mathcal{H}_{bad,\upsilon}) = \mathrm{End}(\pi_{\upsilon,E}^{K_\upsilon^{\mathbf{P}}})$ where

$$\mathcal{H}_{bad,\upsilon} := C_c^\infty(K_\upsilon \setminus \mathbf{G}(F_\upsilon^+)/K_\upsilon, E). \tag{4.34}$$

We then have that $\mathrm{Im}(\mathcal{H}_{bad,\upsilon}) \otimes \mathcal{O}_\Omega = \mathrm{End}(\pi_{\upsilon,E}^{K_\upsilon^{\mathbf{P}}}) \otimes \mathcal{O}_\Omega \subseteq \mathrm{End}(M_p(\mathcal{D}_\Omega^{\mathbf{P}}, K_\upsilon^{\mathbf{P}}))_{\tilde{\pi}}^{\leq h}$. Specialising this at υ yields

$$\mathrm{End}(\pi_{\upsilon,E}^{K_\upsilon^{\mathbf{P}}}) \otimes E_\upsilon \cong \mathrm{Im}(\mathcal{H}_{bad,\upsilon}) \otimes E_\upsilon \subseteq \mathrm{End}(\pi_{\upsilon,\upsilon,E_\upsilon}^{K_\upsilon^{\mathbf{P}}}) \tag{4.35}$$

Consequently $(\dim \pi_{\upsilon,E}^{K_\upsilon^{\mathbf{P}}})^2 = \dim \mathrm{End}(\pi_{\upsilon,E}^{K_\upsilon^{\mathbf{P}}}) \leq \dim \mathrm{End}(\pi_{\upsilon,\upsilon,E_\upsilon}^{K_\upsilon^{\mathbf{P}}}) = (\dim \pi_{\upsilon,\upsilon,E_\upsilon}^{K_\upsilon^{\mathbf{P}}})^2$ and equation 4.32 forces the claim. We now construct a canonical family of lines. For $\upsilon \in S$, take an idempotent $e_\upsilon \in \mathrm{End}(\pi_{\upsilon,E}^{K_\upsilon^{\mathbf{P}}})$ such that $e_\upsilon(\pi_{\upsilon,E}^{K_\upsilon^{\mathbf{P}}}) = E\varphi_\upsilon$, and then a Hecke operator $T_\upsilon \in \mathcal{H}_{bad,\upsilon}$ such that $T_\upsilon \to e_\upsilon$. Such Hecke operators on $M_p(\mathcal{D}_\Omega^{\mathbf{P}}, K_f^{\mathbf{P}})_{\tilde{\pi}}^{\leq h}$ satisfy a characteristic polynomial

$$P_\upsilon(x) := (x-1)^{r_\upsilon} x^{d-r_\upsilon} \tag{4.36}$$

where $d = \dim \tilde{\pi}_f^{K_\upsilon^{\mathbf{P}}}$. Hence, there exist $A, B \in \mathcal{O}_\Omega[X]$ such that

$$A(x)(x-1)^{r_\upsilon} + B(x)x^{d-r_\upsilon} \tag{4.37}$$

which allows us to lift e_υ to families as

$$e_\upsilon := B(x)x^{d-r_\upsilon} \tag{4.38}$$

where $\mathrm{Im}(e_\upsilon) = \ker((T_\upsilon - 1)^{r_\upsilon})$. This then cuts out a canonical line

$$L_\Omega := \left(\prod_{\upsilon \in S} e_\upsilon\right)(M_p(\mathcal{D}_\Omega^{\mathbf{P}}, K_p^{\mathbf{P}})) \tag{4.39}$$

such that $\mathrm{sp}_{\underline{\lambda}}(L_\Omega) = Lf_{\underline{\lambda}}$ for $f_{\underline{\lambda}}$ corresponding to the original φ_f. We then pick $f_\Omega \in L_\Omega$ such that $\mathrm{sp}_{\underline{\lambda}}(f_\Omega) = f_{\underline{\lambda}}$. Specialising to a classical $v \in \Omega_{\mathrm{cl}}$, we have $e_v = T_v \in \mathrm{End}(\pi_{v,v}^{K_v})$ and this still cuts out a line $L_{v,v} := e_v(\pi_{v,v}^{K_v^{\mathbf{P}}})$ because the identity $e_v^2 - e_v = 0$ is unaffected and if it does not cut out a line we get a contradiction in terms of dimensions. We claim the Hecke operator has an algebraic eigenvalue. To do this, we first observe

$$L_{v,v} = \ker(T_v - a_{v,v}) = \ker(\mathrm{sp}_v((x-1)^{r_v})) \tag{4.40}$$

for an eigenvalue $a_{v,v} \in L$. However, as it is an eigenvalue, this means $a_{v,v} \in E_v/E$, potentially enlarging $E_v \subseteq L$. Consequently, this means we have a line

$$\ell_{v,v} \subseteq \pi_{v,v,E_v}^{K_v^v} \tag{4.41}$$

such that $\ell_{v,v} \otimes L = L_{v,v}$. We can then pick $\varphi_{v,v} \in L_{v,v}$ such that via lemma 2.2 we have f_v and there exist $c_v \in L$ such that $c_v f_v = \mathrm{sp}_v(f_\Omega)$. □

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